Mathematics-Computer Science 4215H – Mathematical Logic
TRENT UNIVERSITY, Winter 2021

Solutions to Assignment #1
Due on Friday, 22 January.

Do all of the following problems, which are straight out of Chapter 1 of the textbook[0] (which explains the numbering), reproduced here for your convenience with a few explanations in the added footnotes.

1.1. *[Problem 1.1]* Why are the following *not* formulas of $\mathcal{L}_P$? There might be more than one reason ... *[1.5 = 3 × 1.5 each]*

(1) A_{-56}
(3) $A_7 \leftarrow A_4$
(5) $(A_8 A_9 \to A_{1043998}$

SOLUTIONS. (1) Atomic formulas of $\mathcal{L}_P$ only have indices in $\mathbb{N}$ and $-56 \notin \mathbb{N}$.
(2) $\leftarrow$ is not a symbol of $\mathcal{L}_P$. Also, every non-atomic official formula of $\mathcal{L}_P$ is enclosed by parentheses, which the given string of symbols lacks.
(3) Two atomic formulas cannot occur next to each other in a formula; there must always be at least one connective symbol between them. Also, there is no right outside parenthesis to ballance the left outside parenthesis. □

1.5. *[Problem 1.5]* What are the possible lengths[1] of formulas of $\mathcal{L}_P$? Prove it. *[5]*

SOLUTION. Every formula must have at least one symbol, so there are no formulas of length 0.

There exist formulas of length 1: every atomic formula is a formula of length 1.

The only ways to get longer formulas from shorter formulas are to negate a formula – note that $(\neg\alpha)$ adds three symbols to the symbols present in α – or to make an implication between two formulas – note that $(\alpha \to \beta)$ adds three symbols to the symbols present in α and β. It follows that there are no formulas of length 2 or 3, since adding three symbols to a formula or formulas with 1 or more symbols results in a formula with at least 4 symbols.

There exist formulas of lengths 4 and 5, since $(\neg A_0)$ and $(A_1 \to A_2)$, respectively, are examples of such.

There are no formulas of length 6. Suppose, by way of contradiction, that γ is a formula of length 6. Since γ is not atomic, being of length greater than 1, it must either be $(\neg\alpha)$ for some shorter formula α or $(\beta \to \delta)$ for some shorter formulas β and δ. In the former case, α would have to have length 3, which we have already shown to be impossible; in the latter case, one β and δ would have to have a combined length of 3, so one would have to have length 1 and the other would have to have length 2, which last we have also shown to be impossible. Either way, there can be no formula of length 6.

[0] *A Problem Course in Mathematical Logic*, Version 1.6.
[1] As sequences of symbols.

There are formula of lengths 7, 8, and 9, since the formulas $(\neg\,(\neg A_0))$, $(\neg\,(A_1 \to A_2))$, and $(A_0 \to (A_1 \to A_2))$, respectively, are examples of such.

There exist formulas of every length greater than or equal to 10. Suppose $n \geq 10$. Then $n = 3k$ for some $k \geq 4$, $n = 3k+1$ for some $k \geq 3$, or $n = 3k+2$ for some $k \geq 3$.

If $n = 3k$ with $k \geq 4$, we can make a formula of length n by negating or formula of length 9 above $k-3$ times, which would result in a formula with $9+3(k-3) = 9+3k-9 = 3k = n$ symbols.

If $n = 3k+1$ with $k \geq 3$, we can make a formula of length n by negating our formula of length 7 above $k-2$ times, which would result in a formula with a total of $7+3(k-2) = 7+3k-6 = 3k+1 = n$ symbols.

If $n = 3k+2$ with $k \geq 3$, we can make a formula of length n by negating our formula of length 8 above $k-2$ times, which would result in a formula with a total of $8+3(k-2) = 8+3k-6 = 3k+2 = n$ symbols.

Combining all of the above, there are formulas of $\mathcal{L}_P$ of length n for all natural numbers n except for 0, 2, 3, and 6. □

1.7. *[Proposition 1.7]* Show that the set of formulas of $\mathcal{L}_P$ is countable[2]. *[5]*

Solution. We will use parts (1) and (4) of Proposition A.1 of Appendix A. Note first that the set of symbols of $\mathcal{L}_P$ can be put into a 1–1 correspondence with the natural numbers,

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 6 & 7 & \dots & k+4 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \dots & \updownarrow & \dots \\ (&) & \neg & \to & A_0 & A_1 & A_2 & \dots & A_k & \dots , \end{array}$$

so it is countable (see Definition A.4). By Proposition A.1(4), it follows that the set of all finite sequences of symbols of $\mathcal{L}_P$ is countable. Since the set of formulas of $\mathcal{L}_P$ is a subset of this set of finite sequences, and since the set of formulas of $\mathcal{L}_P$ is an infinite set since every one of the infinitely many atomic formulas is a formula, it follows by Proposition A.1(1) that the set of formulas of $\mathcal{L}_P$ is countable. □

1.9. *[Problem 1.9]* Write out $((\alpha \lor \beta) \land (\beta \to \alpha))$ using only $\neg$ and $\to$. *[1.5]*

Solution. Starting with the given unofficial (but fully parenthesized) formula,

$$((\alpha \lor \beta) \land (\beta \to \alpha)) \ ,$$

we first replace $(\alpha \lor \beta)$ by its official form $((\neg\alpha) \to \beta)$,

$$(((\neg\alpha) \to \beta) \land (\beta \to \alpha)) \ ,$$

and then replace the construction $(\gamma \land \delta)$ by its official form of $(\neg\,(\gamma \to (\neg\delta)))$, where γ is $((\neg\alpha) \to \beta)$ and δ is $(\beta \to \alpha)$:

$$(\neg\,(((\neg\alpha) \to \beta) \to (\neg(\beta \to \alpha)))) \qquad \square$$

[2] That is, all the elements of the set can be put into a list indexed by the natural numbers.

1.11. *[Problem 1.11]* Find all the subformulas of each of the following formulas[3]. *[2 = 2 × 1 each]*

(1) $(\neg((\neg A_{56}) \to A_{56}))$

(2) $A_9 \to A_8 \to \neg(A_{78} \to \neg\neg A_0)$

SOLUTIONS. (1) Note that the given formula is already in official form.

$(\neg((\neg A_{56}) \to A_{56}))$ is a subformula of itself.

Stripping away the outermost connective, a $\neg$, of the given formula, we see that $((\neg A_{56}) \to A_{56})$ is also a a subformula of $(\neg((\neg A_{56}) \to A_{56}))$.

Stripping away the outermost connective of $((\neg A_{56}) \to A_{56})$, the $\to$, we see that $(\neg A_{56})$ and A_{56} are subformulas of $((\neg A_{56}) \to A_{56})$, and hence of $(\neg((\neg A_{56}) \to A_{56}))$.

Stripping away the outermost connective of $(\neg A_{56})$, the $\neg$, we see that A_{56} is a subformula of $(\neg A_{56})$, and hence of $(\neg((\neg A_{56}) \to A_{56}))$.

A_{56}, which we have reached twice in taking the given formula apart, is an atomic formula and so cannot be taken apart.

It follows that the set of subformulas of $(\neg((\neg A_{56}) \to A_{56}))$ is:

$$\{\,(\neg((\neg A_{56}) \to A_{56}))\,,\ ((\neg A_{56}) \to A_{56})\,,\ (\neg A_{56})\,,\ A_{56}\,\}$$

(2) We first put $A_9 \to A_8 \to \neg(A_{78} \to \neg\neg A_0)$ into official form, using the grouping and other conventions, to get $(A_9 \to (A_8 \to (\neg(A_{78} \to (\neg(\neg A_0))))))$.

$(A_9 \to (A_8 \to (\neg(A_{78} \to (\neg(\neg A_0))))))$ is a subformula of itself.

Stripping away the outermost connective, a $\to$, of the given formula, we see that A_9 and $(A_8 \to (\neg(A_{78} \to (\neg(\neg A_0)))))$ are subformulas of the given formula.

A_9 is an atomic formula and so cannot be taken apart.

Stripping away the outermost connective of $(A_8 \to (\neg(A_{78} \to (\neg(\neg A_0)))))$, another $\to$, we see that A_8 and $(\neg(A_{78} \to (\neg(\neg A_0))))$ are also subformulas of the given formula.

A_8 is an atomic formula and so cannot be taken apart.

Stripping away the outermost connective of $(\neg(A_{78} \to (\neg(\neg A_0))))$, a $\neg$, we see that $(A_{78} \to (\neg(\neg A_0)))$ is also a subformula of the given formula.

Stripping away the outermost connective of $(A_{78} \to (\neg(\neg A_0)))$, a $\to$, we see that A_{78} and $(\neg(\neg A_0))$ are also subformulas of the given formula.

A_{78} is an atomic formula and so cannot be taken apart.

Stripping away the outermost connective of $(\neg(\neg A_0))$, a $\neg$, we see that $(\neg A_0)$ is also a subformula of the given formula.

Stripping away the outermost connective of $(\neg A_0)$, another $\neg$, we see that A_0 is also a subformula of the given formula.

A_0 is an atomic formula and so cannot be taken apart.

It follows that the set of subformulas of $A_9 \to A_8 \to \neg(A_{78} \to \neg\neg A_0)$ is:

$$\begin{gathered}\{\,(A_9 \to (A_8 \to (\neg(A_{78} \to (\neg(\neg A_0))))))\,,\ (A_8 \to (\neg(A_{78} \to (\neg(\neg A_0)))))\\ (\neg(A_{78} \to (\neg(\neg A_0))))\,,\ (\neg(\neg A_0))\,,\ (\neg A_0)\,,\ A_{78},\ A_9,\ A_8,\ A_0\,\}\qquad\square\end{gathered}$$

[Total = 15]

[3] That is, of their official versions.