Mathematics-Computer Science 4215H – Mathematical Logic
TRENT UNIVERSITY, Winter 2021

Assignment #5
Due on Friday, 26 February.

Do all of the following problems, two of which are straight out of the textbook[0] (which explains the numbering), reproduced here for your convenience.

4.12. *[Theorem 4.12 – Completeness Theorem]* If Δ is a set of formulas and α is a formula such that $\Delta \models \alpha$, then $\Delta \vdash \alpha$. *[5]*

4.13. *[Theorem 4.13 – Compactness Theorem]* A set of formulas Γ is satisfiable if and only if every finite subset of Γ is satisfiable. *[4]*

In proving the above results, you may appeal to any preceding results and problems in the textbook that you like. Given that, both should be fairly easy to put away. The following application of the Compactness Theorem (Hint!), will require probably greater effort.

RT. *[Ramsey's Theorem]* For every integer $n > 0$ there is an integer $R_n > 0$ such that if $G = (V, E)$ is a graph with at least R_n vertices, then G has a clique of size n or an independent set of size n. *[6]*

This problem requires some background, some of which you have probably seen elsewhere:

- A *graph* G is a pair (V, E) consisting of a set V of *vertices* and set $E \subset V \times V$ of *edges* such that $(u, v) \in E \iff (v, u) \in E$.
- A *clique* of a graph $G = (V, E)$ is a subset $C \subseteq V$ of the vertices such that for all u, $v \in C$, $(u, v) \in E$.
- An *independent set* of a $G = (V, E)$ is a subset $I \subseteq V$ of the vertices such that for all u, $v \in I$, $(u, v) \notin E$.
- *[Infinite Ramsey's Theorem]* A graph $G = (V, E)$ with infinitely many vertices must have an infinite clique or an infinite independent set. [You may, and will probably need to, assume this theorem.]

Your task over Reading Week is to try to figure out how the Compactness Theorem could be useful in proving Ramsey's Theorem. I will be forthcoming with hints *after* Reading Week ...

[Total = 15]

[0] *A Problem Course in Mathematical Logic*, Version 1.6.