MATH 4215H

Applications of the Compactness Theorem (for first-order logic)

Compactness Theorem: A set of sentences Σ of a first-order language $\mathcal{L}$ is satisfiable if and only if every finite subset of Σ is satisfiable.

2 Applications: 1) Showing that certain infinite structures exist.

2) Construct "non-standard" models of familiar structures.

Example: Show there exists an infinite graph.

Our language has only one two place relation, E, which tells us if two elements of the universe (ie two vertices) have an edge between them.

Our set Σ_1 of sentences will include

It's an ordinary graph!

$$\forall x \forall y \, (Exy \rightarrow Eyx)$$ (ie the graph is bidirectional)

$$\forall x \, (\neg Exx)$$ (ie the graph has no loops)

plus the following: For each $n \geq 2$ a sentence which asserts that there are at least n things:

$$\exists x_1 \ldots \exists x_n \, ((\neg x_1 = x_2) \wedge (\neg(x_1 = x_3)) \wedge \ldots \wedge (\neg x_1 = x_n) \wedge (\neg x_2 = x_3) \wedge \ldots \wedge (\neg x_2 = x_n) \ldots \wedge (\neg x_{n-1} = x_n)).$$

If we have finite graphs of arbitrarily large size, we can satisfy any finite subcollection of the sentences in Σ_1. Then the Compactness Thm. says we can satisfy all of them at once, ie there is an infinite graph.

Example: A non-standard model of arithmetic.

Pick a language for dealing with the natural numbers, including at least:

S - successor fn., $0, 1,$ $+, \cdot, <$

Use whatever axioms you want to make $S, 0, 1, +, \cdot, <$ work as they should (eg Peano axioms for arithmetic). Include these in our set Σ of sentences, plus the following sentences

$$\exists x\, (\underbrace{S(S \ldots (S}_{n\ S's}(0)) \ldots) < x) \qquad \text{for each } n \geq 1.$$

Every finite subset of Σ is satisfiable in the usual structure $\mathcal{N} = (\mathbb{N}, 0_{\mathbb{N}}, 1_{\mathbb{N}}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, S_{\mathbb{N}}, <_{\mathbb{N}})$.

The Compactness Theorem says that all of the sentences can be satisfied at once, but this requires having an "integer" x s.t. $n < x$ for all $n \in \mathbb{N}$.