MATH 4215H

Completeness of First-Order Logic VI – Satisfaction in the structure continued.

We were in the middle of proving

(*) $\cancel{\mathfrak{A}}\ \mathfrak{M} \models \varphi[s] \iff \Sigma_1 \vdash \varphi^{x_1,\dots,x_k}_{c_1,\dots,c_k}$ where $x_1,\dots,x_k$ are all the free variables of φ & each c_i is given by $[c_i]_\sim = s(x_i)$

$\iff \varphi^{x_1,\dots,x_k}_{c_1,\dots,c_k} \in \Sigma_1$

Σ_1 maximally consistent set with set ~~of~~ witnesses C, & $\mathfrak{M}$ constructed as previously described.

by induction on n = # connectives & quantifiers in φ...

I.H.: $(n \leq m)$ Assume that $\mathfrak{M} \models \varphi[s] \iff \Sigma_1 \vdash \varphi^{x_1,\dots,x_k}_{c_1,\dots,c_k}$ for all φ with n connectives & quantifiers, & all assignment $s: V \to |\mathfrak{M}|$

I.S.: 3 cases: (φ has $m+1$ connectives & quantifiers)

(1) φ is $(\neg\alpha)$ for some α

(2) φ is $(\alpha \to \beta)$ — " — α & β

(3) φ is $\forall x \alpha$ — " — α

(1) $\mathfrak{M} \models \varphi [s] \iff \mathfrak{M} \models \neg \alpha [s]$ since φ is $\neg\alpha$.

$\iff$ not $\mathfrak{M} \models \alpha [s]$ but α has $m < m+1$ connectives, so

$\iff$ not $\Sigma_1 \vdash \alpha^{x_1 \dots x_k}_{c_1 \dots c_k}$ by IH.

$\iff \Sigma_1 \vdash \neg\alpha^{x_1 \dots x_k}_{c_1 \dots c_k}$ since Σ_1 is max. cons.

$\iff \Sigma_1 \vdash \varphi^{x_1 \dots x_k}_{c_1 \dots c_k}$ since φ is $\neg\alpha$

(2) $\mathfrak{M} \models \varphi [s] \iff \mathfrak{M} \models (\alpha \to \beta) [s]$ since φ is $\alpha \to \beta$

$\iff$ either $\mathfrak{M} \not\models \alpha [s]$ or $\mathfrak{M} \models \beta [s]$ α & β each have $\leq m$ connectives & grant.

$\iff$ either $\Sigma_1 \nvdash \alpha^{x_1 \dots x_k}_{c_1 \dots c_k}$ or $\Sigma_1 \vdash \beta^{x_1 \dots x_k}_{c_1 \dots c_k}$

$\iff$ ~~either~~ $\Sigma_1 \vdash (\alpha \to \beta)^{x_1 \dots x_k}_{c_1 \dots c_k}$ since Σ_1 max. cons.

$\iff \Sigma_1 \vdash \varphi^{x_1 \dots x_k}_{c_1 \dots c_k}$.

(3) $\mathfrak{M} \models \varphi [s] \iff \mathfrak{M} \models \forall x \alpha [s]$ since φ is $\forall x \alpha$.

$\iff \mathfrak{M} \models \alpha [s(x | [c]_\sim)]$

$\iff \Sigma_1 \vdash \alpha^{x_1 \dots x_k, x}_{c_1, \dots, c_k, c}$ since $s(x|[c]_\sim)$ is an assignment since x is (maybe) free in α

$\iff \Sigma_1 \vdash \forall x \alpha^{x_1 \dots x_k}_{c_1 \dots c_k}$ using a variation of Gen. on Constants

$\iff \Sigma_1 \vdash \varphi^{x_1 \dots x_k}_{c_1 \dots c_k}$ since φ is $\forall x \alpha$. //

How does this help us prove that

$$\mathfrak{M} \models \sigma \iff \sigma \in \Sigma_1 \quad \text{for all sentences } \sigma \in \Sigma_1 ?$$

(since we wanted to show that Σ_1 is satisfiable ...)

Well: $\mathfrak{M} \models \sigma \iff \mathfrak{M} \models \sigma[s]$ for every assignment s

$\iff \mathfrak{M} \models \sigma[s]$ for some assignment s since σ is a sentence

$\iff \Sigma_1 \vdash \sigma^{x_1, \dots, x_k}_{c_1, \dots, c_k}$ where $x_1, \dots, x_k$ are the variables of σ and $s(x_i) = [c_i]_\sim$

$\iff \Sigma_1 \vdash \sigma$ but σ has <u>no</u> free variables, so $\sigma^{x_1, \dots, x_k}_{c_1, \dots, c_k}$ <u>is</u> σ.

$\iff \sigma \in \Sigma_1$ since Σ_1 is max. cons.

~~Thm 8:~~

Corollary: (Completeness Theorem) If Γ is a consistent set of sentences in a first-order language $\mathcal{L}$, then Γ is satisfiable.

proof: Given such a Γ & $\mathcal{L}$, add a future set of witnesses C to $\mathcal{L}$ to make $\mathcal{L}'$ & expand Γ to a maximally consistent set Σ, with C as a set of witnesses in $\mathcal{L}'$. We then know that Σ is satisfiable by a structure $\mathfrak{M}'$ of $\mathcal{L}'$, and since $\Gamma \subseteq \Sigma$ we know that Γ is also satisfied by $\mathfrak{M}'$! But now if we define a structure $\mathfrak{M}$ for $\mathcal{L}$ by discarding the interpretations of the symbols in C, we have $\mathfrak{M} \models \Gamma$ (in $\mathcal{L}$), so Γ is satisfiable. //

Thus a set of sentences in a first-order language is consistent $\Leftrightarrow$ satisfiable.

<u>Corollary</u>: (Compactness Theorem)
If every finite subset of a set of sentences is satisfiable, then the entire set of set of sentences is satisfiable, and <u>vice versa</u>.

<u>proof</u>: Just like in propositional logic. //

Next time: A brief look at some uses of the Compactness Thm. of first-order logic.