MATH 4215H

Completeness of First-Order Logic V - Satisfaction in the structure

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Thm: Suppose Σ is a maximally consistent set of sentences and C is a set of witnesses for Σ in the first-order language $\mathcal{L}$. Then there is a structure $\mathfrak{M}$ for $\mathcal{L}$ s.t. $\mathfrak{M} \models \Sigma$.

pf: So far, we've defined $\mathfrak{M}$ as follows:

Define an equivalence relation $\sim$ on C by $c \sim d \iff \Sigma \vdash c=d \iff c=d \in \Sigma$.

Then the universe of $\mathfrak{M}$ is $|\mathfrak{M}| = \{ [c]_\sim \mid c \in C \}$.

For each constant symbol c of $\mathcal{L}$,

$$c^{\mathfrak{M}} = \begin{cases} [c]_\sim & \text{if } c \in C \\ [d]_\sim & \text{for some } d \in C \text{ s.t. } c=d \in \Sigma \text{ if } c \notin C. \end{cases}$$

For each k-place function symbol f of $\mathcal{L}$,

$$f^{\mathcal{M}}([c_1]_\sim, \ldots, [c_k]) = [c]_\sim$$

for a $c \in C$ s.t. $\Sigma_1 \vdash c = fc_1 \ldots c_k$

$$(\text{ie } c = fc_1 \ldots c_k \in \Sigma_1.)$$

For each k-place relation symbol of $\mathcal{L}$,

$P^{\mathcal{M}} \subseteq |\mathcal{M}|^k$ is given by

$$P^{\mathcal{M}} = \{([c_1]_\sim, \ldots, [c_k]_\sim) \in |\mathcal{M}|^k \mid \Sigma_1 \vdash Pc_1 \ldots c_k\}$$

$$(\text{ie } Pc_1 \ldots c_k \in \Sigma_1)$$

⌈Checking that $\mathcal{M}$ is well-defined, ie that the definitions of $f^{\mathcal{M}}$, $c^{\mathcal{M}}$, $P^{\mathcal{M}}$ don't depend on the particular representatives chosen from each equivalence class is left to the interested reader.⌋

We need to show that for any formula φ of $\mathcal{L}$ and assignment $s : V \to |\mathfrak{M}|$, we have

$$(\ast) \quad \Sigma_1 \models \varphi[s] \iff \Sigma_1 \vdash \varphi^{x_1, \dots, x_n}_{c_1, \dots, c_n}$$

$$\iff \varphi^{x_1, \dots, x_n}_{c_1, \dots, c_n} \in \Sigma_1$$

where $x_1, \dots, x_n$ are all the free variables of φ & c_i is given by $s(x_i) = [c_i]_\sim$.

<u>Lemma:</u> If $s : V \to |\mathfrak{M}|$ is an assignment and f is a k-place fn. symbol of $\mathcal{L}$ and $t_1, \dots, t_k$ are terms of $\mathcal{L}$, then

$$\bar{s}(f t_1 \dots t_k) = f^{\mathfrak{M}}(\underbrace{\bar{s}(t_1)}_{=[c_1]_\sim}, \dots, \underbrace{\bar{s}(t_k)}_{=[c_k]_\sim})$$

(by defn.)

$$= [c]_\sim$$

s.t. $\Sigma_1 \vdash c = f c_1 \dots c_k$ (i.e. $c = f c_1 \dots c_k \in \Sigma_1$).

<u>pf:</u> By defn... //

We'll prove (*) by induction on how formulas are built, i.e. on the number n of quantifiers &/or connectives in φ.

<u>Base Step:</u> $(n=0)$ φ has no connectives or quantifiers, so it is atomic, i.e. φ is (1) $t_1 = t_2$ for some terms t_1 & t_2 or (2) φ is $Pt_1 \dots t_k$ for some k-place relation symbol P of $\mathcal{L}$.

(1) $\Sigma_1 \models t_1 = t_2\,[s] \iff \bar{s}(t_1) = \bar{s}(t_k) = [c]_\sim$ for some $c \in C$

$\iff \Sigma_1 \vdash c = t_1{}^{x_1,\dots,x_k}_{c_1,\dots,c_n}$ for the variables $x_1,\dots,x_n$ occurring in t_1, & t_2 & constants $c_1,\dots,c_n \in C$

& $\Sigma_1 \vdash c = t_2{}^{x_1,\dots,x_n}_{c_1,\dots,c_n}$

$\iff \Sigma_1 \vdash (t_1 = t_2)^{x_1 \dots x_n}_{c_1 \dots c_n}$ [This follows from the lemma] ✓

(2) Similarly for $Pt_1 \dots t_k$, ...

Inductive Hypothesis: ($n \leq m$) Assume that $\Sigma_1 \models \varphi\, [s]$

$$\iff \Sigma_1 \vdash \varphi^{x_1 \ldots x_k}_{c_1 \ldots c_k}$$

for all formulas φ with $n \leq m$ ~~free variables~~ connectives & quantifiers

& all assignments $s : V \to |\mathfrak{M}|$.

Inductive Step: ($n = m+1$) Suppose φ has $m+1$ connectives & / or quantifiers.

Three cases:
(1) φ is $(\neg \alpha)$ for some α
(2) φ is $(\alpha \to \beta)$ — " — α & β
(3) φ is $\forall x \alpha$ — " — α.

More next time!