MATH 4215H

Completeness of First-Order Logic IV - Defining the structure

Thm: (8.14) Suppose that Σ_1 is a maximally consistent set of sentences in some first-order language $\mathcal{L}$ and C is a set of witnesses for Σ_1 in $\mathcal{L}$. Then there is a structure $\mathfrak{M}$ for $\mathcal{L}$ s.t. $\mathfrak{M} \models \Sigma_1$.

proof: First, define an equivalence relation $\sim$ on C by $c \sim d \iff \Sigma_1 \vdash c = d$ [$\iff c = d \in \Sigma_1$ by Prop. 8.7 since $c = d$ is a sentence.]

We need to check that $\sim$ is really an equivalence relation on C:

(a) $\sim$ is reflexive: ie $c \sim c$ for all $c \in C$

1.	$\forall x\ x = x$	(A7) [generalized]
2.	$(\forall x\ x = x) \to c = c$	(A4) [since c is substitutable for x in $x = x$.]
3.	$c = c$	1,2 MP

Thus $\Sigma_1 \vdash c = c$, so $c \sim c$. [and $c = c \in \Sigma_1$].

$\Rightarrow \Sigma_1 \vdash c=d \Leftrightarrow c=d \in \Sigma_1$

(b) $\sim$ is commutative: i.e. $c \sim d \Rightarrow d \sim c$ (for all $c,d \in C$)

1.	$\forall x \forall y\, (x=y \rightarrow (\underset{\alpha}{x=x} \rightarrow \underset{\alpha'}{y=x}))$	(A8) [generalized]
2.	—— " —————— $\rightarrow \forall y\,(c=y \rightarrow (c=c \rightarrow y=c))$	(A4) since c is substitutable for x in $\forall y(x=y \rightarrow (x=x \rightarrow y=x))$
3.	$\forall y\,(c=y \rightarrow (c=c \rightarrow y=c))$	1, 2 MP
4.	—— " —————— $\rightarrow (c=d \rightarrow (c=c \rightarrow d=c))$	(A4) since c is subst. for y in $c=y \rightarrow (c=c \rightarrow y=c)$
5.	$c=d \rightarrow (c=c \rightarrow d=c)$	3, 4 MP
6.	$c=d$	Premiss (in Σ_1)
7.	$c=c \rightarrow d=c$	5, 6 MP
8.	$c=c$	Premiss (in Σ_1) [by (a)]
9.	$d=c$	7, 8 MP

$\therefore\ \Sigma_1 \vdash d=c$ [i.e. $d=c \in \Sigma_1$], so $d \sim c$.

(c) $\sim$ is transitive: ie ($c \sim d$ and $d \sim e$) $\Rightarrow$ $c \sim e$ (for all $c, d, e \in C$)

Exercise: not unlike (b), except more so.

Having defined the equivalence relation $\sim$ on C, let

$$[c]_\sim = \{d \in C \mid c \sim d\} = \{d \in C \mid \Sigma_1 \vdash c = d\}$$
$$= \{d \in C \mid c = d \in \Sigma_1\}$$

be the $\sim$-equivalence class of $c \in C$. Note that we have either $[c]_\sim = [d]_\sim$ or $[c]_\sim \cap [d]_\sim = \emptyset$ for any $c, d \in C$.

We can now define the structure $\mathfrak{M}$ that will satisfy Σ_1:

(1) The universe of $\mathfrak{M}$ will be $|\mathfrak{M}| = \{[c]_\sim \mid c \in C\}$.

ie The elements of the universe of $\mathfrak{M}$ are the $\sim$-equivalence classes of C.

(2) (a) For every constant symbol $c \in C$,

let $c^{\mathfrak{M}} = [c]_{\sim}$.

(b) For every constant symbol d in $\mathcal{L}$ s.t. $d \notin C$,

let $d^{\mathfrak{M}} = [c]_{\sim}$ for any $c \in C$ s.t. $\Sigma_1 \vdash c = d$.

Why is there a $c \in C$ s.t. $\Sigma_1 \vdash c = d$?

Since C is a set of witnesses, there is some $c \in C$

$$\Sigma_1 \vdash \exists x\,(x = d) \to c = d.$$

But $\Sigma_1 \vdash \exists x\,(x = d)$ [Exercise] {Hint: use A7, A8}

so $\Sigma_1 \vdash c = d$.

(3) For every k-place function symbol f of $\mathcal{L}$, we define $f^{\mathfrak{M}} : |\mathfrak{M}|^k \to |\mathfrak{M}|$ as follows:

$$f^{\mathfrak{M}}([c_1]_{\sim}, [c_2]_{\sim}, \ldots, [c_k]_{\sim}) = [c]_{\sim}$$

for any $c \in C$ s.t. $\Sigma_1 \vdash f c_1 c_2 \ldots c_k = c$. Why is there such a $c \in C$? Since C is a set of witnesses, there is a $c \in C$ $\Sigma_1 \vdash \exists x\,(f c_1 \ldots c_k = x) \to f c_1 \ldots c_k = c$

[Similar exercise to the one above.]

(4) For every k-place relation symbol P of $\mathcal{L}$, we define

$$P^{\mathfrak{M}} \subseteq |\mathfrak{M}|^k \quad \text{by}$$

$$([c_1]_\sim, \ldots, [c_k]_\sim) \in P^{\mathfrak{M}}$$

if and only if $\Sigma_1 \vdash P c_1 \ldots c_k$.

This defines $\mathfrak{M}$. We still need to show that $\mathfrak{M} \models \Sigma_1$. Next time!