MATH 4215H

Completeness of First-Order Logic III - Building a structure from witnesses

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Recall: If Σ_1 is a set of sentences and C is a set of constants of a first-order language $\mathcal{L}$, the C is a <u>set of witnesses</u> for Σ_1 in $\mathcal{L}$ if

for every formula φ with at most one free variable x, there is a $c \in C$ s.t. $\Sigma_1 \vdash \exists x\, \varphi \to \varphi^x_c$.

<u>Idea</u>: If Σ_1 proves that something exists that satisfies φ, i.e. $\Sigma_1 \vdash \exists x\, \varphi$, then the $c \in C$ names that something, i.e. witnesses that it exists.

Key Theorem: (left to you) Suppose Γ is a consistent set of sentences in $\mathcal{L}$. Let C be an infinite set of constant symbols <u>not</u> in $\mathcal{L}$, and let $\mathcal{L}' = \mathcal{L} \cup C$. Then there is a maximally consistent set Σ_1 s.t. $\Gamma \subseteq \Sigma_1$ and C is a set of witnesses for Σ_1 in $\mathcal{L}'$.

The other Key Theorem is the following

<u>Thm:</u> Suppose Σ is a maximally consistent set and C is a set of witnesses for Σ in $\mathcal{L}'$. Then there is a structure $\mathfrak{M}$ for $\mathcal{L}'$ s.t. $\mathfrak{M} \models \Sigma$.

<u>proof:</u> 2 stages: 1) construct $\mathfrak{M}$

2) verify that $\mathfrak{M} \models \Sigma$.

1) Recall that C is a set of witnesses for Σ if for all formulas φ of $\mathcal{L}'$ with at most one free variable x, we have $\Sigma \vdash \exists x \varphi \rightarrow \varphi^x_c$.

Define an equivalence relation $\sim$ on C as follows:

For $c, d \in C$, $c \sim d$ if and only if $\Sigma \vdash c = d$.

Why is this an equivalence relation?

a) $\sim$ is reflexive: (i.e. $c \sim c$ for all $c \in C$) [To show $\Sigma_1 \vdash c=c$]

1. $x = x$ (A7)

Thus $\Sigma_1 \vdash x = x$. By the Generalization Thm.,

$\Sigma_1 \vdash \forall x\, x=x$ [also, any generalization of an axiom is an axiom.

Then

1. $\forall x\, x=x$ (A7)[generalization]
2. $(\forall x\, x=x) \rightarrow c=c$ (A4) Since c is substitutable for x in $x=x$
3. $c=c$ 1,2 MP.

b) $\sim$ is commutative: (i.e. $c \sim d \Rightarrow d \sim c$) [To show: $\Sigma_1 \vdash c=d \rightarrow d=c$]

~~1. $x=x \rightarrow (x=x$~~

More next time...