MATH 4215H 2021-03-26

Completeness of First-Order Logic II - Sets of witnesses

(or, why we bothered with extension languages)

Problem: Given a consistent set of sentences Γ in a first-order language $\mathcal{L}$, construct a structure $\mathfrak{M}$ for $\mathcal{L}$ such that $\mathfrak{M} \models \Gamma$.

In propositional logic we were able to construct a suitable truth assignment (the counterpart of a structure) from a maximally consistent Σ s.t. $\Gamma \subseteq \Sigma$. Here, a set which is only maximally consistent does not carry enough information to construct a structure, partly because most languages $\mathcal{L}$ are not rich enough to specify all we need.

We will expand the language $\mathcal{L}$ to a richer language to be able to build a structure.

Defn: Suppose Σ is a set of sentences and C is a set of (some of) the constant symbols of $\mathcal{L}$. Then C is a set of witnesses for Σ in $\mathcal{L}$ if for every formula φ of $\mathcal{L}$ with at most one free variable x, there is a constant symbol $c \in C$ s.t. $\Sigma \vdash \exists x \varphi \to \varphi^x_c$.

Idea: C is a set of witnesses for Σ in $\mathcal{L}$ if C has a name in it for every element of the universe that Σ proves must exist.

Note that C may have multiple names (ie constant symbols) naming the same thing.

[Use: Use these names to build the universe of the structure.]

Proposition: (8.11) Suppose Γ is a set of sentences of $\mathcal{L}$ and C is a set of witnesses for Γ in $\mathcal{L}$. If $\Gamma \subseteq \Sigma$ for any other set of sentences Σ of $\mathcal{L}$, then C is also a set of witnesses for Σ in $\mathcal{L}$.

Lemma: Suppose Σ is a set of sentences, φ is any formula and x is any variable. Then $\Sigma \vdash \varphi$ iff $\Sigma \vdash \forall x \varphi$.

Key Theorem: Suppose Γ is a consistent set of sentences of $\mathcal{L}$. Let C be an infinite (countable) collection of constant symbols that are not in $\mathcal{L}$, and let $\mathcal{L}'$ be the language including everything $\mathcal{L}$ plus the constant symbols in C. Then there is a maximally consistent set Σ in $\mathcal{L}'$ such that $\Gamma \subseteq \Sigma$ and C is a set of witnesses for Σ in $\mathcal{L}'$.

More next time: -