MATH 4215H Completeness of First-Order Logic – Preliminaries
[Ch. 8 in text]

Fact: The true counterpart of "formula" in propositional logic, especially when it comes to truth & falsity, for 1st-order logic is "sentence". Similarly, the true counterpart of truth assignments in propositional logic are the structures of 1st-order logic.

Recall from Thm 6.15:

If σ & τ are sentences of a 1st-order language $\mathcal{L}$ and $\mathfrak{M}$ is a structure for $\mathcal{L}$, then

(1) $\mathfrak{M} \models \neg\sigma \iff \mathfrak{M} \not\models \sigma$

(2) $\mathfrak{M} \models \sigma \to \tau \iff$ if $\mathfrak{M} \models \sigma$, then $\mathfrak{M} \models \tau$

(6) If x is any variable, then

$$\mathfrak{M} \models \forall x\, \sigma \iff \mathfrak{M} \models \sigma.$$

ie "$\models$" behaves like it does for prop. logic.

A lot of the basic results and definitions involving "$\models$" therefore carry over...

<u>Soundness Theorem</u> If σ is a sentence & Γ is a set of sentences of a first-order language $\mathcal{L}$, then

if $\Gamma \vdash \sigma$, then $\Gamma \models \sigma$.

<u>proof:</u> Left to you – very similar, given the results of Chapters 6&7, to the proof for propositional logic. //

<u>Defn:</u> A set of sentences Σ of a first-order language $\mathcal{L}$ is <u>inconsistent</u> if $\Sigma \vdash \neg(\alpha \to \alpha)$ for some formula α of $\mathcal{L}$. Σ is <u>consistent</u> otherwise.

<u>Prop:</u> If a set of sentences of a 1st-order language $\mathcal{L}$ is satisfiable, then it is consistent.

<u>proof:</u> Very similar...//

Some basic facts with the same proof as in propositional logic include:

<u>Prop:</u> If Σ is an inconsistent set of sentences, then $\Sigma \vdash \varphi$ for any formula φ.

<u>Prop:</u> A set of sentences is consistent, if and only if every (finite) subset of it is consistent.

<u>Defn:</u> A set of sentences Σ of a first-order language $\mathcal{L}$ is <u>maximally consistent</u> if Σ is consistent but $\Sigma \cup \{\tau\}$ is inconsistent for any sentence τ of $\mathcal{L}$ s.t. $\tau \notin \Sigma$.

<u>Prop:</u> If $\mathcal{M}$ is a structure for a 1st-order language $\mathcal{L}$, then $Th(\mathcal{M}) = \{\sigma \mid \sigma \text{ is a sentence of } \mathcal{L} \ \& \ \mathcal{M} \models \sigma\}$ is maximally consistent.

Prop.: If Σ_Γ is a maximally consistent set of sentences of a 1^{st}-order language $\mathcal{L}$, and $\Sigma_\Gamma \vdash \tau$ for some sentence τ of $\mathcal{L}$, then $\tau \in \Sigma_\Gamma$.

Similarly, $\Sigma_\Gamma \vdash \neg\tau$ iff $\tau \notin \Sigma_\Gamma$,
& $\Sigma_\Gamma \vdash \tau \rightarrow \rho$ iff $\tau \notin \Sigma_\Gamma$ or $\rho \in \Sigma_\Gamma$.

Thm.: If Γ is a consistent set of sentences of a 1^{st}-order language $\mathcal{L}$, then there is a maximally consistent set Σ_Γ of sentences of $\mathcal{L}$, such that $\Gamma \subseteq \Sigma_\Gamma$.

Unfortunately, a set of sentences needs additional features besides maximal consistency to have enough information to "conveniently" construct a structure that satisfies it. Dealing with this will occupy a fair bit of time... Starting next time.