MATH 4215H

Deductions in first-order logic III – Differences from propositional logic

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A lot is the same:

- same rule of procedure (MP)
- A1 – A3 are the same
- a deduction is still a finite sequence of formulas

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

s.t. each α_i is an axiom (Note: We do have additional axiom schema.)
or a premiss [from whatever set of premisses]
or follows from some preceding formulas in the deduction by MP.

- We still use the notation $\Sigma \vdash \varphi$ ("Σ proves φ") to mean that there is a deduction, using only formulas from Σ as premisses, whose last formula is φ.
($\emptyset \vdash \varphi$ is often written as $\vdash \varphi$)

Similarly, $\Sigma \vdash \Gamma$ means that $\Sigma \vdash \gamma$ for every formula $\gamma \in \Gamma$.

Thus all the basic results about deductions carry over to first-order logic. (See Chapters 3 & 7 and compare) In particular, the Deduction Theorem is still true and basically has the same proof.

One significant additional shortcut is:

<u>Generalization Theorem</u>: Suppose we have a first-order language $\mathcal{L}$, x a variable, Γ a set of formulas of $\mathcal{L}$ s.t. x does <u>not</u> occur free in any formula of Γ, and suppose φ is a formula of $\mathcal{L}$. Then,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Gamma \vdash \forall x\, \varphi.$$

③

proof: We will proceed by induction on the length n of the deduction of φ from Γ.

Base step: $(n=1)$ If $\Gamma \vdash \varphi$ by a deduction of length 1, then φ is the deduction, so φ is there either because (i) it is an axiom
or (ii) it is a premiss, ie $\varphi \in \Gamma$.

(i) If φ is an axiom, then $\forall x \varphi$ is also an axiom, since any generalization of an axiom is also an axiom, by def'n. But the $\forall x \varphi$ is also a deduction, so $\Gamma \vdash \forall x \varphi$.

(ii) If $\varphi \in \Gamma$, then x does not occur free in φ.

Then

1.	φ	Premiss
2.	$\varphi \rightarrow \forall x \varphi$	(A6) since x does not occur free in φ
3.	$\forall x \varphi$	1,2 MP

is a deduction of $\forall x \varphi$ from Γ, so $\Gamma \vdash \forall x \varphi$.

Inductive Hypothesis: ($\leq n$) If $\Gamma \vdash \varphi$ via a shortest deduction of length $\leq n$, then $\Gamma \vdash \forall x \varphi$. (4)

Inductive Step: Suppose $\Gamma \vdash \varphi$ via a shortest deduction of length $n+1$, say $\alpha_1, \alpha_2, \ldots, \alpha_n, \underset{\varphi}{\underset{\parallel}{\alpha_{n+1}}}$.

Since this is the shortest deduction of φ from Γ, α_{n+1}, i.e. φ, must follow from α_i & α_j for some $i, j \leq n$, by MP. Suppose α_j is $\alpha_i \to \varphi$. Since $\alpha_1, \ldots, \alpha_i$ and $\alpha_1, \ldots, \alpha_j$ are deductions of α_i & α_j from Γ, it follows by the I.H. that

$$\Gamma \vdash \forall x \alpha_i \quad \text{and} \quad \Gamma \vdash \forall x (\underbrace{\alpha_i \to \varphi}_{\alpha_j}), \text{ say}$$

via the deductions $\beta_1, \ldots, \beta_k$ & $\delta_1, \ldots, \delta_m$, resp..

We will construct a deduction of φ from Γ as follows:

1	β_1		
⋮	⋮		
k	β_k	$[\underline{i\geq}\ \forall x \alpha_i]$	
$k+1$	δ_1		
⋮	⋮		
$m+k$	δ_m	$[\underline{i\geq}\ \forall x(\alpha_i \to \varphi)]$	
$m+k+1$	$\forall x(\alpha_i \to \varphi) \to (\forall x \alpha_i \to \forall x \varphi)$		[A5]
$m+k+2$	$\forall x \alpha_i \to \forall x \varphi$		$m+k, m+k+1$ MP
$m+k+3$	$\forall x \varphi$		$k+1, m+k+2$ MP

Every formula in this finite sequence is either an axiom, a premiss (from Γ) or follows from preceding formulas by MP.

Thus $\Gamma \vdash \forall x \varphi$, as desired. //

Generalization on Constants Theorem.

Suppose $\mathcal{L}$ is a first-order language, c is a constant of $\mathcal{L}$, Γ is a set of formulas of $\mathcal{L}$ in which c <u>never</u> occurs and φ is a formula of $\mathcal{L}$. Then

if $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall x\, \varphi^c_x$

for some ~~formula~~ variable x which does not occur in φ and where φ^c_x simply replaces all occurrences of c with x. Moreover, there is a deduction of $\forall x\, \varphi^c_x$ from Γ in which c does not occur.

<u>Note:</u> Occasionally useful, but with a more sensitive and intricate proof than the regular Generalization Theorem.

<u>Notation:</u> Analogously with $Th(\mathfrak{M}) = \{\sigma \mid \sigma \text{ is a sentence } \& \ \mathfrak{M} \models \sigma\}$, we have $Th(\Sigma) = \{\tau \mid \tau \text{ is a sentence } \& \ \Sigma \vdash \tau\}$.

set of sentences