MATH 4215H

Deductions in first-order logic – The axiom schema (Chapter 7!)

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Deductions in first-order logic work a lot like those in propositional logic:

One rule of procedure: <u>Modus Ponens</u> ie from $\alpha \to \beta$ and α, deduce β.

We also recycle the axiom schema from propositional logic:

A1: $\alpha \to (\beta \to \alpha)$ (for every formulas α & β)

A2: $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ (-"- α, β, γ)

A3: $((\neg \beta) \to (\neg \alpha)) \to (((\neg \beta) \to \alpha) \to \beta)$ (-"- α, β)

... but we need additional axiom schema to handle our official quantifier $\forall$ and the special relation of $=$.

to handle $\forall$

A4: $(\forall x \alpha) \to \alpha^x_t$ — where α^x_t is α with the occurrences of x controlled the $\forall x$ in $\forall x \alpha$ replaced by the term t, so long as no variable in t gets captured by a quantifier in α^x_t.

A5: $(\forall x (\alpha \to \beta)) \to ((\forall x \alpha) \to (\forall x (\beta)))$

A6: $\alpha \to \forall x \alpha$ — if x does not occur free in α.

to handle $=$

A7: $x = x$ — (for every variable x)

A8: $(x = y) \to (\alpha \to \beta)$ — if α is atomic and β is α with some (possibly all or none) occurrences of x in α replaced by y. (for all variables x & y)

<u>Note</u>: This is a minimalist setup of axioms & rule of procedure, optimized to prove things about, not to prove things in.

We need various technical definitions to setup A4 correctly (& safely!). In particular, we have have to define "substitutability" for terms replacing variables in a formula.

__Def'n:__ Suppose x is a variable, t is a term, & α is a formula of a first-order language $\mathcal{L}$. Then __t is substitutable for x in α__ is defined as follows:

(1) If α is atomic, then t is substitutable for x in α and α^x_t is α with every occurrence of x replaced by t.

__eg__ α is $y = x$ & t is fcz

then α is $y = fcz$.

(2) If α is $(\neg\beta)$, then t is substitutable for x in α if t is substitutable for x in β, and α^x_t is $(\neg\beta^x_t)$.

(3) If α is $(\beta \to \gamma)$, then t is substitutable for x in α iff it is substitutable for x in β and in γ, and α^x_t is $(\beta^x_t \to \gamma^x_t)$. (4)

(4) If α is $\forall \cancel{x} \beta$, then t is substitutable for x in α iff either (a) x does not occur free in α ~~and~~

or (b) y does not occur in t & t is substitutable for x in β.

If so, then ~~In case (a)~~ α^x_t is $\forall y \beta$ (ie just α) if x is y,

and α^x_t is $\forall y \beta^x_t$ if x is not y.

The logical axioms of first-order logic (in a language $\mathcal{L}$) are all instances of the schema (A1) – (A8) and all generalizations of these scheme

($\forall x \ldots \forall z \, \varphi$ is a generalization of φ.

More next time.