MATH 4215H Structures for first-order languages VI – Extensions, theories, and axioms

Proposition 6.17: Suppose $\mathcal{L}$ is a first-order language and $\mathcal{L}'$ is an extension of it. Then a set of formulas Σ of $\mathcal{L}$ is satisfiable in $\mathcal{L}$ iff it is satisfiable in $\mathcal{L}'$.

(Recall that if $\mathcal{L}'$ is an extension of $\mathcal{L}$, then every formula of $\mathcal{L}$ is a formula of $\mathcal{L}'$.)

proof: $\boxed{\Rightarrow}$ Suppose $\mathfrak{M}$ is a structure for $\mathcal{L}$ satisfying Σ on some assignment $s: V \to |\mathfrak{M}|$, ie $\mathfrak{M} \models \Sigma[s]$.

Extend $\mathfrak{M}$ to make a structure $\mathfrak{M}'$ for $\mathcal{L}'$ as follows:

1) $|\mathfrak{M}'| = |\mathfrak{M}|$ ie $\mathfrak{M}'$ uses the same universe as $\mathfrak{M}$.

2) For every constant symbol c of $\mathcal{L}'$, let $c^{\mathfrak{M}'} = c^{\mathfrak{M}}$ if c is in $\mathcal{L}$ and $c^{\mathfrak{M}'} \in |\mathfrak{M}'|$ otherwise
arbitrarily chosen.

3) For every k-place function symbol f of $\mathcal{L}'$, let
$f^{\mathfrak{M}'} = f^{\mathfrak{M}}$ if f is a k-place function symbo of $\mathcal{L}$
& $f^{\mathfrak{M}'}$ is some arbitrary k-place function on $|\mathfrak{M}'|$ otherwise.

4) For every k-place relation symbol P of $\mathcal{L}'$, let
$P^{\mathfrak{M}'} = P^{\mathfrak{M}}$ is P is a k-place relation symbol in $\mathcal{L}$
and $P^{\mathfrak{M}'}$ is some arbitrary k-place relation on $|\mathfrak{M}'|$ otherwise.

We can then show that $\mathfrak{M}' \models \sigma[s] \iff \mathfrak{M} \models \sigma[s]$ for every formula σ of $\mathcal{L}$. [By induction on how terms & formulas are interpreted in a structure, ie an induction on how terms and formulas are built.]
Essentially, $\mathfrak{M}$ & $\mathfrak{M}'$ are identical except for things that don't occur in σ since σ is a formula of $\mathcal{L}$.

$\boxed{\Leftarrow}$ Suppose $\mathfrak{M}'$ is a structure for $\mathcal{L}'$ satisfying Σ on assignment s, ie $\mathfrak{M}' \models \sigma[s]$ for all $\sigma \in \Sigma$,

set of formulas of $\mathcal{L}$

We cut down $\mathcal{M}'$ to be a structure for $\mathcal{L}$, by simply discarding the interpretations of all the symbols in $\mathcal{L}'$ but not $\mathcal{L}$ and keeping the interpretations of the ones in $\mathcal{L}$ unchanged.

Again, we can show that $\mathcal{M}' \models \sigma[s] \iff \mathcal{M} \models \sigma[s]$ for all the formulas σ of $\mathcal{L}$. [Same induction ...]

//

<u>Defn:</u> (1) If $\mathcal{M}$ is a structure for $\mathcal{L}$, then the <u>theory</u> of $\mathcal{M}$ is $Th(\mathcal{M}) = \{\tau \mid \tau$ is a sentence of $\mathcal{L}$ s.t. $\mathcal{M} \models \tau\}$.

(2) If Γ is a set of sentences and $\mathcal{Q}$ is a collection of structures (all for a first-order language $\mathcal{L}$), then Γ is a set of <u>(non-logical) axioms</u> for $\mathcal{Q}$, if $\mathcal{M} \in \mathcal{Q} \iff \mathcal{M} \models \Gamma$.

* Note that we ~~may~~ have different structures $\mathcal{M}$ and $\mathcal{N}$ for a language s.t. $Th(\mathcal{M}) = Th(\mathcal{N})$

<u>ie</u> $\mathcal{M} \models \sigma \iff \mathcal{N} \models \sigma$ for all sentences σ of $\mathcal{L}$.