MATH 4215H Structures for First-Order Languages V – Satisfaction and (semantic) implication

Recap: (1) If $\mathcal{L}$ is a first-order language, φ a formula of $\mathcal{L}$, $\mathfrak{M}$ is a structure for $\mathcal{L}$, and $s: V \to |\mathfrak{M}|$ is an assignment, then $\mathfrak{M} \models \varphi[s]$ means that φ is true about the structure when each variable that is free in φ is assigned the value in $|\mathfrak{M}|$ that s gives it.

(2) $\mathfrak{M} \models \varphi$ ("$\mathfrak{M}$ satisfies φ" or "φ is true in $\mathfrak{M}$") means that $\mathfrak{M} \models \varphi[s]$ for all assignments $s: V \to |\mathfrak{M}|$.

Def'n 6.6: If Σ is a set of formulas of $\mathcal{L}$ and φ is a formula of $\mathcal{L}$, then $\Sigma \models \varphi$ ("Σ entails φ" or "Σ implies φ") if and only if for all structures $\mathfrak{M}$ for $\mathcal{L}$ we have: If $\mathfrak{M} \models \Sigma$, then $\mathfrak{M} \models \varphi$.

[ie, if for all assignment $s: V \to |\mathfrak{M}|$, $\mathfrak{M} \models \Sigma[s]$, then
— " ———————, $\mathfrak{M} \models \varphi[s]$.]

②

Similarly, $\Sigma_1 \models \Gamma$, where Γ is also a set of formulas of $\mathcal{L}$, if for all structures $\mathfrak{M}$ for $\mathcal{L}$, we have: if $\mathfrak{M} \models \Sigma_1$, then $\mathfrak{M} \models \Gamma$.

Notation: We usually write $\models \varphi$ for $\emptyset \models \varphi$.

Defn: (6.7) A formula is a tautology if $\models \varphi$, i.e. $\mathfrak{M} \models \varphi$ for every structure for the language.
A formula is a contradiction if $\models \neg\varphi$ is a tautology.
[This is not the same as $\not\models \varphi$ ...]

Proposition 6.10: Suppose α & β are formulas of a first-order language $\mathcal{L}$. Then $\{(\alpha \to \beta), \alpha\} \models \beta$.

proof: $\{(\alpha \to \beta), \alpha\} \models \beta$ means that for every structure $\mathfrak{M}$ for $\mathcal{L}$, we have that if $\mathfrak{M} \models \{(\alpha \to \beta), \alpha\}$, then $\mathfrak{M} \models \beta$.

This, in turn, means that for every structure $\mathfrak{M}$ for $\mathcal{L}$, if $\mathfrak{M} \models \{(\alpha \to \beta), \alpha\} [s]$ for all assignments $s : V \to |\mathfrak{M}|$,

then $\mathfrak{M} \models \beta [s]$ for all assignments $s : V \to |\mathfrak{M}|$.

Suppose, then, that

$\mathfrak{M} \models \{(\alpha \to \beta), \alpha\} [s]$ for all assignments $s : V \to |\mathfrak{M}|$

Then $\mathfrak{M} \underset{\text{II}}{\models} (\alpha \to \beta) [s]$ and $\mathfrak{M} \models \alpha [s]$ (—— " ——)

$\mathfrak{M} \models \beta [s]$ if $\mathfrak{M} \models \alpha [s]$

so $\mathfrak{M} \models \beta [s]$ for all assignments

as desired.

$\therefore \{(\alpha \to \beta), \alpha\} \models \beta$ //

<u>Prop. 6.11</u>: Suppose Σ_1 is a set of formulas of the first-order language $\mathcal{L}_1$ and φ, η are ~~a~~ formulas of $\mathcal{L}$.

Then $\Sigma_1 \cup \{\varphi\} \models \eta$

if and only if

$\Sigma_1 \models (\varphi \to \eta)$.

<u>proof</u>: Unwind the definitions - left to you. //

<u>Problem 6.14</u>: A set of formulas Σ_1 of a first-order language $\mathcal{L}$ is satisfiable [in some structure on some assignment] <u>iff</u> there is no contradiction χ such that $\Sigma_1 \models \chi$.

<u>proof</u>: Unwind those definitions ... //

There are serious advantages to sticking with sentences because we don't have to worry about particular assignments that may or may not satisfy the sentence.

i.e. We know that σ being a sentence means that if $\mathfrak{M} \models \sigma\,[s]$ for any assignment then $\mathfrak{M} \models \sigma\,[r]$ for all assignments r

i.e. $\mathfrak{M} \models \sigma$.

This means that we can get Prop 6.15 - "$\models$ acts like it did for formulas of prop. logic when dealing with sentences in first-order logic."

<u>Prop</u>: (6.15) Suppose $\mathfrak{M}$ is a structure for a first order language $\mathcal{L}$ and σ & τ are sentences of $\mathcal{L}$. (no free variables!)

Then
(1) $\mathfrak{M} \models \neg\sigma \iff \mathfrak{M} \not\models \sigma$

(2) $\mathfrak{M} \models (\sigma \to \tau) \iff \mathfrak{M} \models \tau$ whenever $\mathfrak{M} \models \sigma$

(6) $\mathfrak{M} \models \forall x \sigma \iff \mathfrak{M} \models \sigma$,

<u>proof</u>. Soon on an assignment near you. //