MATH 4215H Structures for First-Order Languages IV - More on assignments and satisfaction

<u>Prop. 6.5:</u> If $\mathfrak{M}$ is a structure for a first-order language $\mathcal{L}$, s is an assignment, x is a variable, and φ is a formula of $\mathcal{L}$, then $\mathfrak{M} \models \exists x\, \varphi\, [s]$

$\iff \mathfrak{M} \models \varphi\, [s(x|m)]$ for some $m \in |\mathfrak{M}|$

Recall: $\exists x\, \varphi$ is an abbreviation for $\neg \forall x (\neg \varphi)$.

<u>proof:</u> $\mathfrak{M} \models \exists x\, \varphi\, [s] \iff \mathfrak{M} \models \neg \forall x (\neg \varphi)\, [s]$

$\iff \mathfrak{M} \not\models \forall x (\neg \varphi)\, [s]$

$\iff \mathfrak{M} \not\models \neg \varphi\, [s(x|m)]$ for some $m \in |\mathfrak{M}|$

$\iff \mathfrak{M} \models \varphi\, [s(x|m)]$ for some $m \in |\mathfrak{M}|$

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<u>Def'n 6.5</u>: If $\mathfrak{M}$ is a structure for the first-order language $\mathcal{L}$ and φ is a formula of $\mathcal{L}$, then $\mathfrak{M} \models \varphi$ ("$\mathfrak{M}$ satisfies φ" or "$\mathfrak{M}$ makes φ true") <u>iff</u> $\mathfrak{M} \models \varphi\,[s]$ for every assignment $s : V \to |\mathfrak{M}|$.

<u>Aside</u>: $\mathfrak{M} \not\models \varphi$ only means that $\mathfrak{M} \not\models \varphi\,[s]$ for <u>some</u> assignment s. It's possible for this to happen and still have $\mathfrak{M} \models \varphi\,[r]$ on some other assignment.

Similarly, $\mathfrak{M} \models \Sigma$ for a set of formulas Σ if $\mathfrak{M} \models \sigma$ for each $\sigma \in \Sigma$.

If $\mathfrak{M} \models \Sigma$ for some structure $\mathfrak{M}$, then Σ is satisfiable.

Lemma 6.7: If $\mathcal{M}$ is a structure for $\mathcal{L}$, t is a term of $\mathcal{L}$, and r & s are assignments $V \to |\mathcal{M}|$ such that $r(x) = s(x)$ for every variable that occurs in t, then $\bar{r}(t) = \bar{s}(t)$.

proof: Do an induction on how terms are built and extended assignments are defined. //

Prop. 6.8: If $\mathcal{M}$ is a structure for $\mathcal{L}$, φ is a formula of $\mathcal{L}$, and r & s are assignments $V \to |\mathcal{M}|$ such that $r(x) = s(x)$ for every variable x that occurs free in φ, then $\mathcal{M} \models \varphi\,[s] \iff \mathcal{M} \models \varphi\,[r]$.

proof: Do an induction on how formulas are built and satisfaction on an assignment is defined. //

Corollary 6.9. Suppose $\mathfrak{M}$ is a structure for $\mathcal{L}$ and σ is a sentence [a formula with no free variables] of $\mathcal{L}$. Then $\mathfrak{M} \models \sigma$ if and only if $\mathfrak{M} \models \sigma[s]$ for some (any) assignment $s : V \to |\mathfrak{M}|$.

proof: Left to you ... //

- We can use $\models$ for sentences and structures the way we did for formulas and truth assignments in propositional logic.

More on satisfying sentences next time ... (Look at Prop. 6.15 (1)(2))