# MATH 4215H Structures for First-Order Languages III - Interpreting formulas using assignments

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**Recap:** If $\mathcal{M}$ is a structure for a language $\mathcal{L}$, an assignment for this structure is a function

$$s: V \to |\mathcal{M}| \qquad [V = \{v_0, v_1, v_2, \ldots\}]$$

assigning a value in the universe of discourse to each variable symbol in $\mathcal{L}$. The corresponding extended assignment $\bar{s}: T \to |\mathcal{M}|$ [$T$ is the set of terms of $\mathcal{L}$.], given by

(1) For each variable $v_i$, $\bar{s}(v_i) = s(v_i)$.

(2) For each constant $c$ of $\mathcal{L}$, $\bar{s}(c) = c^{\mathcal{M}}$.

(3) For each $k$-place function symbol $f$ of $\mathcal{L}$, and terms $t_1, \ldots, t_k$ (for which $\bar{s}$ has already been defined)

$$\bar{s}(f t_1 \ldots t_k) = f^{\mathcal{M}}(\bar{s}(t_1), \ldots, \bar{s}(t_k)).$$

We can now interpret the truth/falsity of formulas of $\mathcal{L}$ on an assignment $s : V \to |\mathcal{M}|$.

**Defn:** Suppose $\mathcal{M}$ is a structure for the first-order language $\mathcal{L}$, $s : V \to |\mathcal{M}|$ is an assignment, and $\varphi$ is a formula of $\mathcal{L}$. Then $\varphi$ is true in the structure $\mathcal{M}$ on assignment $s$, written as $\mathcal{M} \models \varphi[s]$ ("$\mathcal{M}$ satisfies $\varphi$ on assignment $s$") is defined as follows:

(1) If $\varphi$ is $=t_1 t_2$ (i.e. $t_1 = t_2$) for terms $t_1$ & $t_2$, then $\mathcal{M} \models \varphi[s]$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.

(2) If $\varphi$ is $Pt_1 \dots t_k$ for some $k$-place relation symbol $P$ and terms $t_1, \dots, t_k$, then $\mathcal{M} \models \varphi[s]$ iff $(\bar{s}(t_1), \dots, \bar{s}(t_k)) \in P^{\mathcal{M}}$ i.e. $P^{\mathcal{M}}(\bar{s}(t_1), \dots, \bar{s}(t_k))$ is true.

(3) If $\varphi$ is $(\neg\alpha)$ for some formula $\alpha$, then $\mathcal{M} \models \varphi[s] \iff \mathcal{M} \not\models \alpha[s]$. [i.e. $\mathcal{M}$ does not satisfy $\alpha$ on asst. $s$]

(4) If $\varphi$ is $(\alpha \to \beta)$ for some formulas $\alpha$ & $\beta$, then

$\mathcal{M} \models \varphi[s]$ iff, $\mathcal{M} \models \beta[s]$ whenever $\mathcal{M} \models \alpha[s]$

i.e. unless $\mathcal{M} \models \alpha[s]$ and $\mathcal{M} \not\models \beta[s]$.

(5) If $\varphi$ is $\forall x \alpha$ for some formula $\alpha$, then

$\mathcal{M} \models \varphi[s]$ iff (x is one of the $v_k$) for all $m \in |\mathcal{M}|$, we have

$\mathcal{M} \models \alpha[s(x|m)]$

where $s[(x|m)]$ is the assignment $V \to |\mathcal{M}|$

s.t. $s[(x|m)](v_k) = \begin{cases} s(v_k) & \text{if } v_k \text{ isn't } x \\ m & \text{if } v_k \text{ is } x \end{cases}$.

If $\Sigma$ is a set of formulas of $\mathcal{L}$, then

$\mathcal{M} \models \Sigma[s]$ "$\mathcal{M}$ satisfies $\Sigma$ on assignment $s$"

if $\mathcal{M} \models \sigma[s]$ for all $\sigma \in \Sigma$.

Note: $\mathcal{M} \not\models \Sigma[s]$, this just means that it doesn't satisfy some formula of $\Sigma$, i.e. $\Sigma \not\models \sigma[s]$ for some $\sigma \in \Sigma$.

Example: Let $\mathcal{L}_{\mathbb{Z}}$ be our language for the integers:

constants: $0, 1$

1-place fns: $P, S$ (intended to name the predecessor & successor fns.)

2-place fns: $+, -, \cdot$

relation: $<$

Let $\mathcal{Z} = (\mathbb{Z}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, P_{\mathbb{Z}}, S_{\mathbb{Z}}, +_{\mathbb{Z}}, -_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}})$ be the intended structure for $\mathcal{L}_{\mathbb{Z}}$.

Let $\varphi$ be the formula $\forall v_0 (v_0 = |P v_0 - v_0|)$.

ie $\forall v_0 ( P(v_0) = v_0 - 1 )$

Let $s$ be any assignment for $\mathcal{Z}$. We wade through the definition to figure out if $\mathcal{Z} \models \varphi[s]$.

$\mathbb{Z} \models \varphi\,[s] \iff \mathbb{Z} \models \forall v_0\,(P(v_0) = v_0 - 1)\,[s]$

$\iff \mathbb{Z} \models P(v_0) = v_0 - 1 \quad [s(v_0|z)]$

for every $z \in |\mathbb{Z}| = \mathbb{Z}$.

$\iff P^{\mathbb{Z}}(\bar{s}(v_0|z)(v_0)) = \bar{s}(v_0|z)(v_0) - 1^{\mathbb{Z}}$ for all $z \in \mathbb{Z}$

$\iff P_{\mathbb{Z}}(z) = z - 1_{\mathbb{Z}}$ for all $z \in \mathbb{Z}$

--- which is actually true (of the actual integers ---)

Practice this: look at the examples in the text in Chapter 6 and do Problem 6.4. (Might also try Prop. 6.5)