MATH 4215H — 2021-03-05 — (1)

Structures for First-Order Languages II — Assignments, terms, and formulas.

Defn: Suppose $\mathcal{L}$ is a first-order language and $V = \{v_0, v_1, v_2, \ldots\}$ is its set of variables. An assignment for $\mathcal{L}$ and a structure $\mathfrak{M}$ for $\mathcal{L}$ is a function $s : V \to |\mathfrak{M}|$. ($|\mathfrak{M}|$: universe of $\mathfrak{M}$.)

[Note: this is not a truth assignment.]

Let T be the set of terms of $\mathcal{L}$. Then the extended assignment $\bar{s} : T \to |\mathfrak{M}|$ is defined as follows:

(1) For each variable $\bar{s}(v_k) = s(v_k)$.

(2) For each constant symbol c, $\bar{s}(c) = c^{\mathfrak{M}}$.

(3) For each k-place function symbol f and terms $t_1, \ldots, t_k$ for which $\bar{s}$ has already been defined, $\bar{s}(f t_1 \ldots t_k) = f^{\mathfrak{M}}(\bar{s}(t_1), \ldots, \bar{s}(t_k))$.

Recall our example language $\mathcal{L}_{\mathbb{Z}}$:

constants: 0 & 1

functions: 1-place P & S
2-place $+, \cdot, -$

relation: 2-place $<$

Then $\mathcal{Z} = (\mathbb{Z}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, P_{\mathbb{Z}}, S_{\mathbb{Z}}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, -_{\mathbb{Z}}, <_{\mathbb{Z}})$ is the intended structure for $\mathcal{L}_{\mathbb{Z}}$.

Define $s: V \to \mathbb{Z}$ by $s(v_k) = -k$.

Then if t is the term $\cdot v_k P + v_n 1$,

$$\begin{aligned}
\bar{s}(t) &= \cdot_{\mathbb{Z}} s(v_k) P_{\mathbb{Z}} +_{\mathbb{Z}} s(v_n) 1_{\mathbb{Z}} \\
&= s(v_k) \cdot_{\mathbb{Z}} P_{\mathbb{Z}} (s(v_n) +_{\mathbb{Z}} 1_{\mathbb{Z}}) \\
&= (-k) \cdot_{\mathbb{Z}} P_{\mathbb{Z}} ((-n) +_{\mathbb{Z}} 1_{\mathbb{Z}}) \\
&= -k \cdot ((-n)+1)-1) = (-k)\cdot(-n) = k \cdot n = kn .
\end{aligned}$$

In another structure for the language we'd go through the same steps to evaluate this term. Ditto for starting with a different assignment for $\mathcal{I}$.

Suppose we try this with the assignment $r : V \to \mathbb{Z}$ given by $r(v_k) = 2^k$. Then

$$\begin{aligned}
\bar{r}(t) &= \bar{r}(\circ_{\mathbb{Z}} v_k P_{\mathbb{Z}} v_n 1_{\mathbb{Z}}) \\
&= \circ_{\mathbb{Z}} r(v_k) P_{\mathbb{Z}} r(v_n) 1_{\mathbb{Z}} \\
&= r(v_k) \circ_{\mathbb{Z}} P_{\mathbb{Z}} (r(v_n) + 1_{\mathbb{Z}}) \\
&= 2^k \cdot ((2^n + 1) - 1) \\
&= 2^k \cdot 2^n = 2^{k+n}.
\end{aligned}$$

So the values of terms depend on the particular assignment.

Next time: interpreting formulas