MATH 4215H **Structures for First-Order Languages, or, interpreting formulas** 2021-03-03 ①

Clean-up: (from last time) A <u>sentence</u> is a formula in which there are no free variables.

Recall that $\mathcal{L}_{\mathbb{Z}}$ is the language with the following non-logical symbols: constants: 0 and 1

1-place functions: P and S

2-place functions: $+$, $-$, and $\cdot$

relation: $<$

The intended universe of discourse is $\mathbb{Z}$ with the usual interpretations of the symbols $0, 1, +, -, \cdot, <$ and with P & S naming the predecessor & successor functions, respectively, on $\mathbb{Z}$.

Notation: The intended "structure" is

$$\mathbb{Z} = (\mathbb{Z}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, +_{\mathbb{Z}}, -_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}}, P_{\mathbb{Z}}, S_{\mathbb{Z}}).$$

In general, if $\mathcal{L}$ is a first-order language, a structure $\mathfrak{M}$ for the language consists of the following:

(1) A non-empty set (or "universe") $M = |\mathfrak{M}|$.

(2) For each constant symbol c of $\mathcal{L}$, an element $c_{\mathfrak{M}}$ or $c^{\mathfrak{M}}$ of the universe.

(3) For each k-place function symbol f, a k-place function $f^{\mathfrak{M}}$ (or $f_{\mathfrak{M}}$) $M^k \to M$.

(4) For each k-place relation symbol P, a k-place relation $P^{\mathfrak{M}}$ (or $P_{\mathfrak{M}}$) which is a subset of M^k.

Obvious potential problem: A very different structure can also be a structure for a given language.

Example: Consider the following structure $\mathcal{M}$ for $\mathcal{L}_{\mathbb{Z}}$:

(1) $M = V_\omega$ = collection of all finite sets

(2) $0^{\mathcal{M}} = \emptyset \in M$ and $1^{\mathcal{M}} = \{\emptyset\} \in M$

(3) $P^{\mathcal{M}} : M \to M$ will be the identity function,

i.e. $P^{\mathcal{M}}(m) = m$ for all $m \in M$.

$S^{\mathcal{M}} : M \to M$ will be the function $S^{\mathcal{M}}(m) = m \cup \{m\}$,

$+^{\mathcal{M}} : M^2 \to M$ — " — $+^{\mathcal{M}}(m,n) = m \cup n$

$-^{\mathcal{M}} : M^2 \to M$ — " — $-^{\mathcal{M}}(m,n) = m \setminus n$

$= \{x \in m \mid x \notin n\}$

$\cdot^{\mathcal{M}} : M^2 \to M$ — " — $\cdot^{\mathcal{M}}(m,n) = m \cap n$

(4) $<^{\mathcal{M}} \subseteq M^2$ will be given by $<^{\mathcal{M}}(m,n)$ if and only if $m \subseteq n$.

This structure can be used to interpret formulas of $\mathcal{L}_{\mathbb{Z}}$ just as well as $\mathbb{Z} = (\mathbb{Z}, \ldots)$ can. Obviously, a lot of formulas might have different truth values in these structures. [Next time!]