MATH 4215H First-Order Logic: Free variables, sentences, and extension languages

Recall: We set up a first-order language $\mathcal{L}_{\mathbb{Z}}$ to use as an example. The "non-logical" symbols of $\mathcal{L}_{\mathbb{Z}}$ are:

(1) Constant symbols: 0 and 1
(Intended to name the actual 0 & 1 in $\mathbb{Z}$.)

(2) Function symbols: P & S (1-place)
(Intended to name the predecessor and successor functions.)
Also, $+$, $-$, and $\cdot$. (2-place)
(Intended to name the usual arithmetic operations.)

(3) Relation symbol: $<$ (2-place)
(Intended to name the usual $<$ on $\mathbb{Z}$.)

Observe that a variable in a formula of a first-order language may or may not be controlled by a quantifier.

eg $v_1 = 0$ — No quantifier so v_1 is not controlled. (ie It is "free".)

$\forall v_2 (v_1 = 0)$ — We have a quantifier, but v_1 is not the variable being quantified, so it still isn't being controlled.

$\forall v_1 (v_1 = 0)$ — The quantifier is using v_1, so the occurrence of v_1 is being controlled by the quantifier.

free free "bound" not free

$(v_1 = P(v_2)) \wedge \forall v_1 (v_1 = 0)$ — One occurrence of v_1 is free (as is v_2) and one is not.

free bound by bound by

$\neg \forall v_2 (v_1 = P(v_2)) \wedge \forall v_1 (v_1 = 0)$

Official definition:

Suppose x is a variable of a first-order language.

[So x is one of the variable v_k of the language – we are using "x" as a meta-variable.] no quantifiers!

Then (1) If a formula φ is atomic, then x occurs free in φ if it occurs in φ.

(2) If φ is $(\neg\psi)$, then x occurs free in φ if (and only if) it occurs free in ψ.

(3) If φ is $(\eta \to \zeta)$, then x occurs free in φ if (and only if) it occurs free in at least one of η or ζ.

(4) If φ is $\forall y\,\lambda$ for some formula λ and variable y, then x occurs free in φ if it occurs free in λ and x is not y, and otherwise, if x is y, then any occurrence of x in λ (and hence φ) is bound in φ.

<u>Def'n</u>: If $\mathcal{L}$ is a first-order language, then another first-order language $\mathcal{L}'$ is an <u>extension</u> of $\mathcal{L}$ if every non-logical symbol of $\mathcal{L}$ is a non-logical symbol (of the same type) in $\mathcal{L}'$.

<u>Example</u>: Let's define another language for the integers, call it $\mathcal{L}$:

Non-logical symbols of $\mathcal{L}$ are

constant symbols: 0 and 1

function symbols: $+$, $\cdot$ [both 2-place]

relation symbol: $<$.

Then $\mathcal{L}$ is a "sub-language" of $\mathcal{L}_{\mathbb{Z}}$, and $\mathcal{L}_{\mathbb{Z}}$ is an extension of $\mathcal{L}$.

Note that every formula of $\mathcal{L}$ is also a formula of $\mathcal{L}_{\mathbb{Z}}$. (Ditto for terms!)