

MATH 4215H First-Order Logic: Free variables,
sentences, and extension languages 2021-03-01 ①

Recall: We set up a first-order language $\mathcal{L}_{\mathbb{Z}}$ to use as an example. The "non-logical" symbols of $\mathcal{L}_{\mathbb{Z}}$

are: (1) Constant symbols: 0 and 1

(Intended to name the actual 0 & 1 in \mathbb{Z} .)

(2) Function symbols: P & S (1-place)

(Intended to name the predecessor and successor functions.)

Also, +, -, and \cdot . (2-place)

(Intended to name the usual arithmetic operations.)

(3) Relation symbol: $<$ (2-place)

(Intended to name the usual $<$ on \mathbb{Z} .)

Observe that a variable in a formula of a first-order language may or may not be controlled by a quantifier.

eg $v_1 = 0$

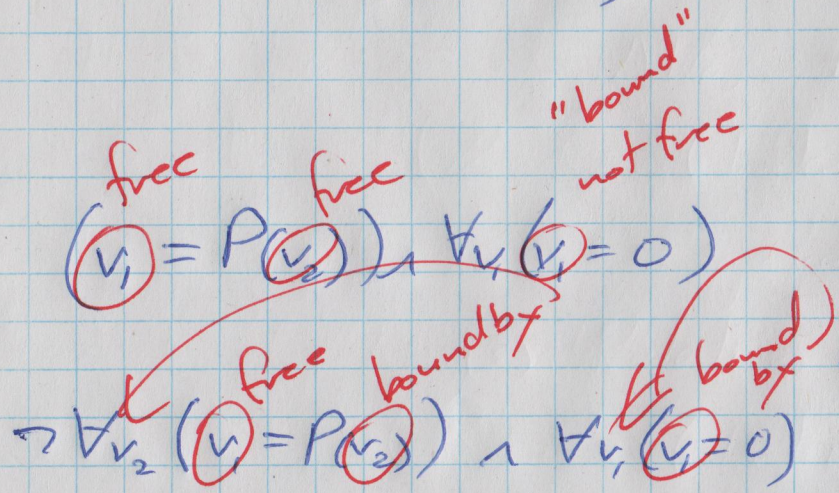
No quantifier, so v_1 is not controlled. (ie It is "free".)

$\forall v_2 (v_1 = 0)$

We have a quantifier, but v_1 is not the variable being quantified, so it still isn't being controlled.

$\forall v_1 (v_1 = 0)$

The quantifier is using v_1 , so the occurrence of v_1 is being controlled by the quantifier.



One occurrence of v_1 is free (as is v_2) and one is not.

Official definition:

(3)

Suppose x is a variable of a first-order language.

[So x is one of the variables v_k of the language -
we are using " x " as a meta-variable.] \rightarrow no quantifiers!

Then (1) If a formula ϕ is atomic, then x occurs free in ϕ if it occurs in ϕ .

(2) If ϕ is $(\neg \psi)$, then x occurs free in ϕ if (and only if) it occurs free in ψ .

(3) If ϕ is $(\mathcal{M} \rightarrow \mathcal{N})$, then x occurs free in ϕ if (and only if) it occurs free in at least one of \mathcal{M} or \mathcal{N} .

(4) If ϕ is $\forall y \mathcal{A}$ for some formula \mathcal{A} and variable y , then x occurs free in ϕ if it occurs free in \mathcal{A} and x is not y , and otherwise, if $x \equiv y$, then any occurrence of x in \mathcal{A} (and hence ϕ) is bound in ϕ .

Def'n: If \mathcal{L} is a first-order language, then another first-order language \mathcal{L}' is an extension of \mathcal{L} if every non-logical symbol of \mathcal{L} is a non-logical symbol (of the same type) in \mathcal{L}' . (4)

Example: Let's define another language for the integers, call it \mathcal{L}_2 :

Non-logical symbols of \mathcal{L}_2 are

constant symbols: 0 and 1

function symbols: +, \cdot [both 2-place]

relation symbol: $<$

Then \mathcal{L} is a "sub-language" of \mathcal{L}_2 , and \mathcal{L}_2 is an extension of \mathcal{L} .

Note that every formula of \mathcal{L} is also a formula of \mathcal{L}_2 .
(Ditto for terms!)