MATH 4215H 2021-02-26 ①

First-Order Logic: An example of a language

We'll set up a language, $\mathcal{L}_{\mathbb{Z}}$, intended to talk about the number system of the integers.
[ie Intended "universe of discourse" for this language $\mathbb{Z}$.]

Symbols:

Common to every first-order language.

(1) Parentheses: (,)

(2) Connectives: $\neg$, $\rightarrow$ [Informally, also $\vee, \wedge, \leftrightarrow$]

(3) Quantifier: $\forall$ [Informally, also $\exists$.] [Short for $\neg\forall\neg$ ...]

(4) Variables: $v_0, v_1, v_2, \ldots$ [Informally, $x, y, z, \ldots$]

(5) Equality: $=$

Optional ("non-logical") symbols chosen for the language on the basis of its intended use.

(6) Constants: 0, 1 [Intended to name the actual 0 & 1 in $\mathbb{Z}$.]

(7) Functions: 1-place fns. P & S [Predecessor & Successor fns.]
2-place fns. $+, -, \cdot$ [~~Also~~ Name the usual arithmetic operations.]

(8) Relations: $<$ [2-place] [Names the usual $<$.]

Before we can define formulas of $\mathcal{L}_{\mathbb{Z}}$, we have to define the <u>terms</u> of $\mathcal{L}_{\mathbb{Z}}$. Terms are expressions built using variables, constants, and/or function symbols, that ought to evaluate out to an element of the universe of discourse. (In this case, an integer.)

The terms of $\mathcal{L}_{\mathbb{Z}}$ are:

(i) Each variable <u>symbol</u> v_n is a term.

(ii) Each constant <u>symbol</u>, in this case 0 & 1, is also a term.

(iii) For each function <u>symbol</u>, following it up with a term for each place of input makes a term. In this case, if t_1 & t_2 are terms of $\mathcal{L}_{\mathbb{Z}}$, then Pt_1, St_1, $+t_1t_2$, $-t_1t_2$, and $\cdot t_1t_2$ are also terms. ["Polish notation" - parentheses free]

(iv) Nothing else is an official term.

eg $+1 \cdot v_1 - 01$ is a term

Informally, we'd write it as something like.

$$1 + [v_1 \cdot (0-1)].$$

$SP1$ is a term

The formulas of $\mathcal{L}_{\mathbb{Z}}$ are:

(i) If t_1 & t_2 are terms, then

$=t_1t_2$ [Informally, $t_1 = t_2$.]

is a formula.

(ii) If t_1 & t_2 are terms, then

$<t_1t_2$ [Informally, $t_1 < t_2$.]

is a formula.

These are the "atomic" formulas of $\mathcal{L}_{\mathbb{Z}}$.

(iii) If α is a formula of $\mathcal{L}_{\mathbb{Z}}$, then so is $(\neg\alpha)$.

(iv) If α & β are formulas of $\mathcal{L}_{\mathbb{Z}}$, then so is $(\alpha \to \beta)$.

(v) If v_n is a variable and δ is a formula of $\mathcal{L}_{\mathbb{Z}}$, then $\forall v_n \delta$ is a formula.

(vi) Nothing else is a formula.

eg $=1SP1$ is a formula (atomic)

[Informally, we'd likely write it as $1 = S(P(1))$.]

<01 is a formula (atomic) [Informally, $0<1$.]

$\forall v_3 \, (<v_3 S v_3)$ is a formula [Informally, $\forall v_3 \, (v_3 < S(v_3))$.]

$\neg \forall v_5 \neg (= P v_5 \, v_5)$ — " — [Informally, $\exists v_5 \, (P v_5 = v_5)$.]

& so on

One thing to watch for later: The language $\mathcal{L}_{\mathbb{Z}}$ could be used in its intended use case to talk about the structure $(\mathbb{Z}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, P_{\mathbb{Z}}, S_{\mathbb{Z}}, +_{\mathbb{Z}}, -_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}})$; but it could be used for any structure that supplies interpretations for each of the symbols, eg

$$(\mathbb{R}, \pi, e, \sin, \cos, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, -_{\mathbb{R}}, >_{\mathbb{R}}),$$

which may have little or nothing to do with the intended use.