MATH 4215H Consistency IV, wherein consistency implies satisfiability

Theorem 4.11: A set of formulas Γ of $\mathcal{L}_p$ is satisfiable, if and only if it is consistent.

proof: $\boxed{\Rightarrow}$ This is Prop. 4.2, done a while back.

$\boxed{\Leftarrow}$ Suppose Γ is a consistent set of formulas.

By 4.10, there is a maximally consistent set Σ_1 such that $\Gamma \subseteq \Sigma_1$. We will define a truth assignment u as follows: let $u(A_n) = T \iff A_n \in \Sigma_1$.

(so $u(A_n) = F \iff A_n \notin \Sigma_1$)

Claim: u satisfies Σ_1 (and hence $\Gamma \subseteq \Sigma_1$)

By induction on the number of connectives, c, in the formula σ.

Base Step: ($c=0$) If σ has no connectives, then σ is an atomic formula, say A_k. Then

$$u(\sigma) = u(A_k) = T \iff A_k \in \Sigma_1 \quad \text{(by defn of } u\text{)}$$
$$\iff \sigma \in \Sigma_1.$$

Inductive Hypothesis: Suppose $u(\sigma) = T \iff \sigma \in \Sigma_1$ for all formulas σ of $\mathcal{L}_p$ with $\leq c$ connectives.

Inductive Step: Suppose σ is a formula of $\mathcal{L}_p$ with $c+1$ connectives. Then (i) σ is $(\neg\alpha)$ for some formula α with c connectives, or (ii) σ is $(\beta \to \delta)$ for some formulas β and δ which have c connectives between them.

case (i): $u(\sigma) = u(\neg\alpha) = T \iff u(\alpha) = F \iff \alpha \notin \Sigma_1$ (I.H.)

$$\iff \neg\alpha \in \Sigma_1 \ (4.8) \iff \sigma \in \Sigma_1$$

case (ii): $u(\sigma) = u(\beta \to \delta) = T \iff u(\beta) = F$ or $u(\delta) = T$

$\iff \beta \notin \Sigma_1$ or $\delta \in \Sigma_1$ (I.H.)

$\iff (\beta \to \delta) \in \Sigma_1$ (4.9) $\iff \sigma \in \Sigma_1$. //

Consequences (coming soon to another assignment):

Completeness Theorem

If Γ is a set of formulas and α is a formula, such that $\Gamma \models \alpha$, then $\Gamma \vdash \alpha$.

Compactness Theorem

A set of formulas is satisfiable, if and only if ~~...~~ every finite subset of it is satisfiable.