MATH 4215H Consistency III – Expanding to Maximal Consistency

Theorem 4.10: If Γ is a consistent set of formulas of $\mathcal{L}_P$, then there is a maximally consistent set of formulas of $\mathcal{L}_P$, Σ, such that $\Gamma \subseteq \Sigma$.

[Recap: Γ is consistent if you can't prove a contradiction from Γ, and Σ is maximally consistent if it is consistent and $\Sigma \cup \{\alpha\}$ is inconsistent for any $\alpha \notin \Sigma$.]

proof: We know that the set of formulas of $\mathcal{L}_P$ is countable, ie the formulas (all of them) can be enumerated indexed by the natural numbers, say $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$

We will inductively construct sets of formulas $\Sigma_0, \Sigma_1, \ldots$ as follows:

$\Sigma_0 = \Gamma$ Note that this is consistent by hypothesis.

Given that Σ_n has been defined for some $n \geq 0$, let

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{g_n\} & \text{if this is consistent} \\ \Sigma_n & \text{otherwise} \end{cases}.$$

Notice that each Σ_n is consistent by the construction. (Also, $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$)

Let $\Sigma = \bigcup_{n \geq 0} \Sigma_n$. Obviously, $\Gamma = \Sigma_0 \subseteq \Sigma$.

Claim that Σ is maximally consistent.

(i) <u>Σ is consistent</u>: If not, there is some finite subset Δ of Σ which is inconsistent (Prop. 4.4 ~~Corollary 4.5~~).

But any finite subset Δ of Σ must be entirely present in some Σ_n [n = maximum k s.t. $g_k \in \Delta$]. This would mean Σ_n is inconsistent [by Cor. 4.5], contradicting it's definition.

(ii) Σ_1 is maximally consistent: Suppose $\alpha \notin \Sigma_1$.
Then α is φ_n for some n, which means that we did not put it into Σ_{n+1} because otherwise it would be in Σ_1. The only reason not to put it in at stage n would be if $\Sigma_n \cup \{\varphi_n\} = \Sigma_n \cup \{\alpha\}$ was inconsistent, i.e. $\Sigma_n \cup \{\alpha\} \vdash \neg(\beta \to \beta)$ for some formula β

Since $\Sigma_n \cup \{\alpha\} \subseteq \Sigma_1 \cup \{\alpha\}$, it ~~would~~ follows that $\Sigma_1 \cup \{\alpha\} \vdash \neg(\beta \to \beta)$, so $\Sigma_1 \cup \{\alpha\}$ is inconsistent.

//

Use: Given a consistent Γ, expand it to a maximally consistent Σ_1, and use Σ_1 to define a truth assignment satisfying Σ_1, and hence Γ.