MATH 4215H - Consistency II - Maximal Consistency

Quick recap of highlights from last time:

Defn: A set of formulas Σ is inconsistent if $\Sigma \vdash \chi$ for some contradiction χ. Σ is consistent if it is not inconsistent.

Prop. 4.2: If a set of formulas is satisfiable, then it is consistent.

Corollary 4.5: A set of formulas is consistent if and only if every finite subset of it is consistent.

This time:

Defn: A set of formulas Σ is maximally consistent if it is consistent, but for any formula α such that $\alpha \notin \Sigma$, $\Sigma \cup \{\alpha\}$ is inconsistent.

Q.: Are there any maximally consistent sets?

A.: Yes!

Problem 4.6: Suppose u is a truth assignment. Then
$$\Sigma_1 = \{ \varphi \mid u(\varphi) = T \}$$ is maximally consistent.

proof: Σ_1 is satisfiable (since u satisfies it by its definition), so it is consistent by Prop. 4.2.

Suppose $\alpha \notin \Sigma_1$. Then, by the def'n of Σ_1, $u(\alpha) = F$. Then $u(\neg\alpha) = T$, so $\neg\alpha \in \Sigma_1$. But then $\Sigma_1 \cup \{\alpha\}$ includes $\{\alpha, \neg\alpha\}$ as a finite subset, and by 3.9(1), $\{\alpha, \neg\alpha\} \vdash \chi$ for any contradiction χ. Thus $\Sigma_1 \cup \{\alpha\}$ has a finite inconsistent subset, so Σ_1 is inconsistent by Corollary 4.5. //

Prop. 4.7: If Σ_1 is a maximally consistent set of formulas and $\Sigma_1 \vdash \gamma$, then $\gamma \in \Sigma_1$.

proof: Suppose Σ_1 is maximally consistent and $\Sigma_1 \vdash \gamma$.
Assume, by way of contradiction, that $\gamma \notin \Sigma_1$.
$\therefore$ $\Sigma_1 \cup \{\gamma\}$ is inconsistent, so for ~~any~~ some contradiction χ,
$\Sigma_1 \cup \{\gamma\} \vdash \chi$. ~~Since $\Sigma_1 \vdash \gamma$, Σ_1~~

By the Deduction Theorem, it follows that

$$\Sigma_1 \vdash \gamma \rightarrow \chi$$

Since we already have $\Sigma_1 \vdash \gamma$, it follows that $\Sigma_1 \vdash \chi$, so ~~Σ~~ Σ_1 is inconsistent, contradicting Σ_1 being consistent. //

Prop. 4.8: If Σ_1 is maximally consistent and α is a formula, then $\neg\alpha \in \Sigma_1 \iff \alpha \notin \Sigma_1$.

Prop. 4.9: If Σ_1 is maximally consistent and $\alpha \rightarrow \beta$ is a formula, then $\alpha \rightarrow \beta \in \Sigma_1 \iff (\alpha \notin \Sigma_1 \text{ or } \beta \in \Sigma_1)$.

Left to you.

Next time ~~we~~ will prove:

Theorem 4.10: If Γ is a consistent set of formulas, then there is a maximally consistent set of formulas Σ_1 such that $\Gamma \subseteq \Sigma_1$.

Eventually, we will use, given a consistent set Γ ~~an~~ a maximally consistent set Σ_1 containing Γ to define a truth assignment satisfying Γ!