

MATH 4215H Consistency [Chapter 4]

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We will establish the link between deductions and entailment ("truth implication"). We have the easy part already:

Soundness Theorem: Given a set of formulas Σ and a formula α , if $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$.

We will aim to prove the converse:

Completeness Theorem: Given a set of formulas Σ and a formula α , if $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.

Proving this directly is very hard (even if it's possible), but the contrapositive is within reach (with work):

if $\Sigma \not\vdash \alpha$, then $\Sigma \not\models \alpha$.

ie If there is no deduction ~~that~~ of α from Σ , there is a truth assignment satisfying Σ but not α ,

that is there is a truth assignment
satisfying both all of Σ and $\neg\alpha$.

$$\underline{12} \quad \text{---} \quad \text{---} \quad \Sigma \cup \{\neg\alpha\}.$$

Pulling this off means we need to define and work with the counterpart of satisfiability for deductions/provability.

Def'n: A set of formulas Γ is inconsistent if there is a contradiction χ s.t. $\Gamma \vdash \chi$.
To keep things simple, we will use any formula of the form $\neg(\alpha \rightarrow \alpha)$ for χ .

Prop. 4.2: If a set of formulas is satisfiable, then it is consistent.

proof: We prove the contrapositive: if Σ is inconsistent, then Σ is not satisfiable. If $\Sigma \vdash \neg(\alpha \rightarrow \alpha)$, so by the Soundness Thm., $\Sigma \models \neg(\alpha \rightarrow \alpha)$, but no truth assignment can satisfy $\neg(\alpha \rightarrow \alpha)$, so Σ can't be satisfied. //

Strategy: To show that any consistent set of formulas is satisfiable. This requires constructing the necessary truth assignment from the given set of formulas somehow... (3)

Some preliminaries: (soon to be on an assignment... :))

Prop. 4.3: If Σ is an inconsistent set of formulas, then $\Sigma \vdash \phi$ for any formula ϕ .

Prop. 4.4: Suppose Σ is an inconsistent set of formulas. Then there is a finite subset Δ of Σ such that Δ is inconsistent.

Corollary 4.5: A set of formulas Σ is consistent if and only if every finite subset Δ of Σ is consistent.

Next time: "Maximal consistency"