MATH 421~~6~~5H 2021-01-29 ①

Deductions III, or the Deduction Theorem

Deduction Theorem: Suppose Σ_1 is a set of formulas and α and β are any formulas. Then $\Sigma_1 \vdash (\alpha \to \beta)$ if and only if $\Sigma_1 \cup \{\alpha\} \vdash \beta$.

proof: $\boxed{\Longrightarrow}$ Assume $\Sigma_1 \vdash (\alpha \to \beta)$, say by way of the deduction $\delta_1, \delta_2, \ldots, \delta_n$ $(\delta_n \equiv (\alpha \to \beta))$. Then $\delta_1, \delta_2, \ldots, \underbrace{\delta_n}_{\equiv\, \alpha \to \beta}, \alpha, \beta$ is a deduction of β from $\Sigma_1 \cup \{\alpha\}$, by the defn of deduction. //

$\boxed{\Longleftarrow}$ Assume $\Sigma_1 \cup \{\alpha\} \vdash \beta$ via the (shortest) deduction $\gamma_1, \ldots, \gamma_n$ $(\gamma_n \equiv \beta)$. By induction on n, the length of this shortest deduction.

Base Step: $(n = 1)$ So $\beta \in \Sigma_1 \cup \{\alpha\}$ or β is an axiom.

⟨Rest is for you.⟩

Inductive Hypothesis: ($n \le k \ge 1$) For every ~~deduction~~ formula β from $\Sigma \cup \{\alpha\}$ with a shortest deduction $\eta_1, \ldots, \eta_n$, we have that $\Sigma \vdash (\alpha \to \beta)$.

Inductive Step: ($n = k+1$) Suppose β is a formula such there is a (shortest) deduction of β from $\Sigma \cup \{\alpha\}$.

~~such that~~ [To show: $\Sigma \vdash (\alpha \to \beta)$]

Suppose that shortest deduction is

$$\eta_1, \eta_2, \ldots, \eta_k, \underset{\beta}{\underset{\parallel}{\eta_{k+1}}}.$$

Then β must be obtained from preceding η_i & η_j (with $\eta_j \equiv (\eta_i \to \beta)$) by MP.

Then <left to you>. //

[Coming to an assignment soon.]

The Deduction Theorem allows us to take shortcuts [ie work around having only one rule of procedure].

eg $\{\delta, \neg\delta\} \vdash \gamma$ (Problem 3.9(1) in the text)

1. $((\neg\gamma) \to \delta) \to (((\neg\gamma) \to (\neg\delta)) \to \gamma)$ A3
2. $\delta \to ((\neg\gamma) \to \delta)$ A1
3. δ Premiss
4. $(\neg\gamma) \to \delta$ 2,3 MP
5. $[(\neg\gamma) \to (\neg\delta)] \to \gamma$ 1,4 MP
6. $(\neg\delta) \to ((\neg\gamma) \to (\neg\delta))$ A1
7. $\neg\delta$ Premiss
8. $(\neg\gamma) \to (\neg\delta)$ 6,7 MP
9. γ 5,8 MP

Consequence, via the Deduction Thm. is that $\{\delta\} \vdash (\neg\delta) \to \gamma$; applying it again gives $\vdash \delta \to ((\neg\delta) \to \gamma)$.