MATH 4215H Deductions II

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or, proving things about deductions

Recap:

Axiom schema: (i.e. templates for axioms)

(A1) $(\alpha \to (\beta \to \alpha))$ for any formulas α & β

(A2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$ — " — α, β, γ

(A3) $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$ — " — α, β

Rule of Inference: Modus Ponens

"From $\varphi \to \psi$ and φ, infer ψ."

$$\begin{array}{|l} \varphi \to \psi \\ \varphi \\ \hline \psi \end{array} \quad \text{(MP)}$$

A deduction from a set of formulas Σ, [the premisses or hypotheses]

is a finite sequence $\varphi_1\ \varphi_2\ \varphi_3 \ldots \varphi_n$ of formulas

such that for each $k \leq n$ either φ_k is an axiom

or φ_k is a hypothesis [$\in \Sigma$]

or φ_k follows from some φ_i & φ_j (with $i, j < k$) by MP.

Example: $\{(\neg\beta)\to(\neg\alpha), \alpha\} \vdash \beta$

there is a deduction of β from the given set of premisses

1.	$((\neg\beta)\to(\neg\alpha))\to(((\neg\beta)\to\alpha)\to\beta)$	A3
2.	$(\neg\beta)\to(\neg\alpha)$	Premiss
3.	$((\neg\beta)\to\alpha)\to\beta$	1,2 MP
4.	$\alpha\to((\neg\beta)\to\alpha)$	A1
5.	α	Premiss
6.	$(\neg\beta)\to\alpha$	4,5 MP
7.	β	3,6 MP

Prop. 3.5: If $\Gamma \vdash \delta$ and $\Gamma \vdash (\delta \to \beta)$, then $\Gamma \vdash \beta$.

proof: Since $\Gamma \vdash \delta$, there is a deduction of δ from Γ

ie a sequence of formulas ~~such~~

$$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$$

such that each α_i is in Γ or is an axiom or follows from preceding α's by MP, and also α_k is δ.

Since $\Gamma \vdash (\delta \to \beta)$, there is a deduction of $(\delta \to \beta)$ from Γ

ie a sequence of formulas

$$\beta_1, \ldots, \beta_n$$

such that each β_j is in Γ or is an axiom or follows from preceding β_i's by MP, and β_n is $(\delta \to \beta)$.

We need to find or make a deduction of β from Γ.

Consider the sequence of formulas

$$\alpha_1, \alpha_2, \ldots, \underset{\delta}{\underset{\parallel\!\!\mid}{\alpha_k}}, \beta_1, \beta_2, \ldots, \underset{(\delta \to \beta)}{\underset{\parallel\!\!\mid}{\beta_n}}, \cancel{\beta}\ \beta$$

↑ follows from δ & $\delta \to \beta$ by MP

This sequence has the property that every formula is in Γ or is an axiom or follows from preceding formulas by MP. Since the last formula in the sequence is β, this sequence is a deduction of β from Γ, ie $\Gamma \vdash \beta$. //

<u>Theorem 4.1</u> (Soundness Theorem)

If Σ_1 is a set of formulas and α is a formula such that $\Sigma_1 \vdash \alpha$, then $\Sigma_1 \models \alpha$.

<u>proof:</u> We will proceed by induction on the length of the shortest deduction of α from Σ_1.
(All applicable α's at once.)

<u>Base case:</u> ($n=1$) In this case α is the deduction of α from Σ_1. This means that α is (i) in Σ_1 or (ii) it's an axiom.

Suppose u is any truth assignment that makes every formula in Σ_1 true (ie $\forall \sigma \in \Sigma_1 : u(\sigma) = T$).
If α is in Σ_1, then $u(\alpha) = T$ by our assumption that u makes every formula in Σ_1 true.
If α is an axiom, then α is a tautology, so $u(\alpha) = T$. $\therefore \Sigma_1 \models \alpha$.

Induction Hypothesis: Suppose that we have shown that $\Sigma_1 \vdash \alpha$ implies that $\Sigma_1 \models \alpha$ for every α with shortest deduction from Σ_1 having length $n \geq 1$.

Inductive Step: $(n+1)$ Suppose α is a formula with $\Sigma_1 \vdash \alpha$ via a shortest deduction of length $n+1$,

say $\alpha_1, \alpha_2, \ldots, \alpha_n, \underset{\alpha}{\underset{\parallel}{\alpha_{n+1}}}$.

Since $n \geq 1$, $n+1 > 1$, so this shortest deduction of α from Σ_1 has to have used MP at the last step, say α_{n+1} (is α) follows from α_i & $\underset{\alpha_i \to \alpha_{n+1}}{\underset{\parallel}{\alpha_j}}$ by MP for some $i, j \leq n$.

By the I.H., we have $\Sigma_1 \models \alpha_i$ & $\Sigma_1 \models \alpha_i \to \alpha_{n+1}$ (because the sequence $\alpha_1, \ldots, \alpha_i$ is a deduction of α_i from Σ_1 & — " — $\alpha_1, \ldots, \alpha_j$ — " — α_j from Σ_1,) and $i, j \leq n$

But, if $\Sigma_1 \vdash \alpha_i$ & $\Sigma_1 \vdash \alpha_i \to \alpha_{n+1}$, then $\Sigma_1 \vdash \alpha_{n+1}$, 7

and α_{n+1} is our α, so $\Sigma_1 \vdash \alpha$. //