MATH 4215H - Deduction in Propositional Logic

2021-01-25 ①

A <u>deduction</u> is a formal proof in propositional logic.

<u>Keypoint</u>: This is supposed independent of checking things with truth tables and should be compatible with what truth tables do. [really, truth assignments]

$\downarrow$ A sequence of formulas, starting from some hypotheses, that reaches a conclusion that follows logically from the hypotheses. Each formula is a hypothesis, or follows from preceding steps in the proof, or is "logically true".

The "logically true" formulas are the axioms of $\mathcal{L}_P$.

<u>Defn</u>: For any formulas α, β, & γ of $\mathcal{L}_P$, the following are axioms:

<u>A1</u>: $(\alpha \to (\beta \to \alpha))$

<u>A2</u>: $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$

<u>A3</u>: $(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$

Technically, these are <u>axiom schema</u>. Plugging particular α, β, & γ yields an axiom.

Note that every instance of each of the schema (i.e. every axiom) is a tautology. [You can check this with truth tables.] These are the formulas (besides whatever hypotheses a deduction starts from) that we plunk down in the course of the deduction.

The idea that a step in the proof may follow from preceding ones is formalized in the idea of a "rule of inference". We will only have <u>one</u> rule of inference:

<u>Modus Ponens</u>: Given formulas α and $\alpha \rightarrow \beta$, we can infer the formula β. [i.e. implication works].

Note that more practical systems may trade of having axioms for having more rules of procedure. [This one is optimized to prove things about.]

Def'n: Suppose Σ is a set of formulas (the hypotheses) (or premiss).
A deduction from Σ is a sequence of formulas $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n$
such that for each $k \leq n$, we have that

(1) φ_k is an axiom (an instance of A1, A2, or A3),

or (2) $\varphi_k \in \Sigma$ (ie φ_k is a hypothesis), [premiss]

or (3) there are $i, j < k$ such that φ_k follows from φ_i and φ_j by Modus Ponens (MP).

[Note that this means that φ_i is $\varphi_j \to \varphi_k$ or that φ_j is $\varphi_i \to \varphi_k$.]

If there is a deduction $\varphi_1, \ldots, \varphi_n$ of $\varphi = \varphi_n$ from Σ, then we say that "Σ proves φ", usually written as $\Sigma \vdash \varphi$.

"$\vdash \varphi$" means "$\emptyset \vdash \varphi$", ie φ can be proved from the axioms alone.

$\Sigma \vdash \Gamma$ means $\Sigma \vdash \gamma$ for all $\gamma \in \Gamma$.

Let's show that $\{(A_1 \to A_2), (A_2 \to A_3)\} \vdash (A_1 \to A_3)$.

1.	$A_1 \to A_2$	Premiss
2.	$A_2 \to A_3$	Premiss
3.	$\overset{\alpha}{(A_2 \to A_3)} \to (\overset{\beta}{A_1} \to \overset{\alpha}{(A_2 \to A_3)})$	A1
4.	$A_1 \to (A_2 \to A_3)$	2, 3 MP
5.	$(\overset{\alpha}{A_1} \to (\overset{\beta}{A_2} \to \overset{\gamma}{A_3})) \to ((\overset{\alpha}{A_1} \to \overset{\beta}{A_2}) \to (\overset{\alpha}{A_1} \to \overset{\gamma}{A_3}))$	A2
6.	$(A_1 \to A_2) \to (A_1 \to A_3)$	4, 5 MP
7.	$A_1 \to A_3$	1, 6 MP

$\therefore \{(A_1 \to A_2), (A_2 \to A_3)\} \vdash (A_1 \to A_3)$.