MATH-COIS 4215H - Truth Assignments II

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(Chapter 2 cont'd.)

Recap: A truth assignment is a function

$$u: \{A_0, A_1, A_2, \ldots\} \to \{T, F\},$$

which we extend to a function on all formulas of $\mathcal{L}_p$ via (a) For any formula α for which $u(\alpha)$ has been defined, $u(\neg\alpha) = \begin{cases} F & u(\alpha) = T \\ T & u(\alpha) = F. \end{cases}$

and (b) For any formulas α & β for which u has been defined, $u(\alpha \to \beta) = \begin{cases} F & \text{if } v(\alpha) = T \text{ \& } v(\beta) = F \\ T & \text{otherwise}. \end{cases}$

This is a more formal version of the way we use truth tables.

Terminology: If u is a truth assignment and α is a formula of $\mathcal{L}_p$, then "<u>u satisfies α</u>" if $u(\alpha) = T$.

Similarly, if Σ_1 is a collection of formulas then "<u>u satisfies Σ_1</u>" if $u(\sigma) = T$ for every $\sigma \in \Sigma_1$.

(ie u satisfies Σ_1, if u satisfies σ for all $\sigma \in \Sigma_1$)

Further, a formula (or a collection of formulas) is said to be "<u>satisfiable</u>" if there is <u>some</u> truth assignment u such that u satisfies the formula (resp. the collection of formulas).

Eg Let u be the truth assignment s.t. $u(A_n) = T$ for all n. Then u satisfies $A_0 \to A_0$ and the collection $\{A_0, A_1, A_2, \ldots, A_0 \to A_1, \neg(\neg A_0)\}$

but not $\neg A_0$ or the collection $\{\neg A_0, A_1, A_2, \ldots, A_0 \to A_1, \neg(\neg A_0)\}$

not satisfiable since can't make $\neg A_0$ & $\neg(\neg A_0)$ both true on the same assignment

Defn: A formula φ is a tautology if every truth assignment satisfies it; similarly φ is a contradiction if no truth assignment satisfies it.

eg $\alpha \to \alpha$ is a tautology [Check for yourselves]
and $\neg(\alpha \to \alpha)$ is a contradiction.

Prop. 2.2: Suppose α is a formula of $\mathcal{L}_P$ and u and v are truth assignments such that $u(A_n) = v(A_n)$ for every atomic formula A_n which occurs in α. Then $v(\alpha) = u(\alpha)$.

proof: Via an induction on how formulas are built, ie by induction on the number of connectives in the formula. //

Implication via truth assignments:

<u>Def'n:</u> A set of formulas Σ <u>implies</u> (or <u>entails</u>) a formula α if every truth assignment that satisfies Σ also satisfies α.

"double turnstile"

<u>Notation:</u> "Σ implies α" is often written as $\Sigma \vDash \alpha$.

If Σ does not imply α, <u>ie</u> some truth assignment u satisfies Σ but not α, we often write this as $\Sigma \nvDash \alpha$.

Similarly, for sets of formulas Σ & Δ, $\Sigma \vDash \Delta$ (<u>ie</u> "Σ implies Δ") if $\Sigma \vDash \delta$ for <u>every</u> formula $\delta \in \Delta$.

Observations: A formula α is a tautology

iff $\emptyset \models \alpha$

and α is a contradiction

iff $\emptyset \not\models \alpha$.

Prop 2.9: Suppose Σ is a set of formulas and α and β are formulas.

Then $\Sigma \models (\alpha \to \beta)$

iff $\Sigma \cup \{\alpha\} \models \beta$.

Prop 2.10: A set of formulas Σ is satisfiable

iff there is no contradiction χ

such that $\Sigma \models \chi$.