Welcome to MATH-COIS 4215H: Mathematical Logic

We'll be looking at propositional [Part I] & first-order [Part II] logic in detail.

Q: What, exactly, is a proof?

Why do proofs work?

What can't be proven? [beyond the scope of this course, but see Part IV of the textbook]

What is the (a little bit of) the relationship between a mathematical statement and the objects that make it true (or of which it's true)?

One problem: all this requires a fair bit of abstraction (and proofs) and a lot of exacting attention to detail. The details matter!

Propositional (or sentential or predicate) logic

This is the logic that tries to capture the sense of connectives such as "and", "or", "not", "if... then...", ...

The language we'll be using is $\mathcal{L}_P$ ("L sub P").

Def'n: The symbols of $\mathcal{L}_P$ are:

(1) Parentheses: "(" and ")" - grouping symbols (punctuation)

(2) Connectives: "$\neg$" ("not") and "$\rightarrow$" ("implies" or "if..., then...")

We stick to just these two officially, to keep the language as simple as possible to prove things about. Unofficially, we'll use "$\wedge$" ("and") and "$\vee$" ("or") and "$\leftrightarrow$" ("if and only if") as shorthand for certain constructions using $\rightarrow$ & $\neg$.

(3) Atomic Formulas: "A_0", "A_1", "A_2", ...
["A_n" for $n \in \mathbb{N} = \{0, 1, 2, ...\}$]

Informally, we'll often use $A, B, C, ...$ as informal shorthand when the precise atomic formula is not important.

Note: One could get by with an even simpler language – ~~es~~ one could do without any grouping symbols ([Reverse] Polish Notation) or with just one connective symbol (NOR or NAND) – but at the cost of being much less humanly readable.

We also need to specify how the symbols can be assembled into formulas (ie statements in the language).

We'll use lower-case Greek characters for general formulas.

Def'n: The formulas of $\mathcal{L}_P$ are those finite sequences (or strings) of the symbols of $\mathcal{L}_P$ satisfying:

(1) Every atomic formula is a formula.

(2) If α is a formula, then $(\neg\alpha)$ is also a formula.

(3) If α and β are formulas, then $(\alpha \to \beta)$ is also a formula.

(4) Nothing else is a formula of $\mathcal{L}_P$.

[Every formula is built up in finitely many steps using rules (1)-(3).]

Examples: (0) A_{37} is a formula.

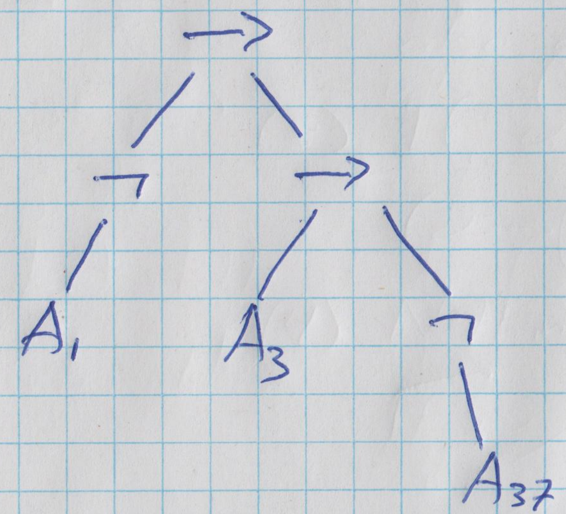
(1) $(\neg A_{13})$ is a formula

(2) $(A_5 \rightarrow A_{23})$ is a formula.

(3) $((\neg A_1) \rightarrow (A_3 \rightarrow (\neg A_{37})))$ is a formula.

<u>etc.</u>

Tree description of (3) above:



Use: Representing statements held together by connectives,

A_0

"Trent is a university

connective and Trent has students".

A_1

This would be represented by
$(A_0 \wedge A_1)$,
which is short for
$(\neg(A_0 \rightarrow (\neg A_1)))$.

<u>Proposition 1.2</u>: Show that every formula of $\mathcal{L}_P$ has the same number of left as right parentheses.

<u>proof 1</u>: (By induction on the length of the formula, as in the number of symbols, counting repetitions, used in the formula.)

<u>Base step</u>: ($n=1$) Suppose the formula α has length 1. Then α is an atomic formula, say A_n. A_n has <u>no</u> parentheses, so it has the same number of left parentheses as right parentheses.

<u>Inductive Hypothesis</u>: ($n=k$, for some $k \geq 1$) If the formula α of $\mathcal{L}_P$ has $\leq k$ symbols, then it has an equal number of left and right parentheses.

Inductive Step: $(n=k+1)$ Suppose α has $k+1$ symbols. Since $k \geq 1$, this means that α has more than one symbol. Thus α had to have built from (a) shorter formula(s). Two cases:

(1) α is $(\neg \beta)$ for some formula β

Since α has $k+1$ symbols, β must have $(k+1)-3 = k-2 < k$ symbols. By I.H., β has the same number of left as right parentheses, say p of each. But then α has $p+1$ left and $p+1$ right parentheses, so it has an equal number of each.

(2) α is $(\beta \rightarrow \gamma)$ for some formulas β and γ.

Since α has $k+1$ symbols, β and γ must have $(k+1)-3 = k-2 < k$ symbols put together and hence less than k each. By the I.H.

, it follows that each of β and γ has the same number of left as right parentheses, say q of each for β and r of each for γ. Then α has $q+r+1$ left and $q+r+1$ right parentheses, so it has an equal number of each.

It follows by induction that every formula of $\mathcal{L}_p$ has the same number of left as right parentheses. //

Next time: the better way to do this – by induction on the number connectives in α.

& then develop a bit more about $\mathcal{L}_p$

& then move on to Chapter 2!