PASCAL

"Essay pour les Coniques"

(Translated from the French by Dr. Frances Marguerite Clarke, Bryn Mawr College, Bryn Mawr, Penna.)

When Pascal (see page 67) was only sixteen years old, he wrote a brief statement which was doubtless intended by him as the first step in an extended study of conics to be undertaken at some future time. In the following year it was printed in the form of a broadside and bore the simple title, "Essay povr les coniqves. Par B. P." Of this single page only two copies are known, one at Hannover among the papers of Leibniz, and the other in the Bibliothèque nationale at Paris. The illustrations here given appeared at the top of the original broadside. The third lemma involves essentially the "Mystic Hexagram" of Pascal. This translation first appeared in *Isis*, X, 33, with a facsimile of the entire essay, and is reproduced in revised form with the consent of the editors.

Essay on Conics

First Definition

When several straight lines meet at the same point, or are parallel to each other, all these lines are said to be of the same order or of the same *ordonnance*, and the totality of these lines is termed an order of lines, or an *ordonnance* of lines.[1]

Definition II

By the expression "conic section," we mean the circle, ellipse, hyperbola, parabola, and an angle; since a cone cut parallel to its base, or through its vertex, or in the three other directions which produce respectively an ellipse, a hyperbola, and a parabola, produces in the conic surface, either the circumference of a circle, or an angle, or an ellipse, a hyperbola, or a parabola.

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[1] [This definition is taken almost word for word from Desargues. See the notes to the Brunschvicg and Boutroux edition, t. I., Paris, 1908. This translation is made from the facsimile of the original as given in this edition, and acknowledgment is hereby made of the assistance rendered by these notes in determining the meaning of several passages. It should also be said that the text of this edition is in marked contrast to the imperfect one given in the Paris edition of 1819.]

Definition III

By the word "droite" (straight) used alone, we mean "ligne droite" (straight line).[1]

Lemma I

If in the plane *M*, *S*, *Q*, two straight lines *MK*, *MV*, are drawn from point *M* and two lines *SK*, *SV* from point *S*; and if *K* be the point of intersection of the lines *MK*, *SK*; *V*, the point of intersection of the lines *MV*, *SV*; *A*, the point of intersection of the lines *MA*, *SA*; and μ, the point of intersection of the lines

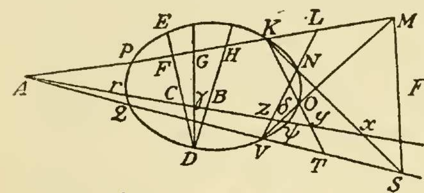

Fig. 1

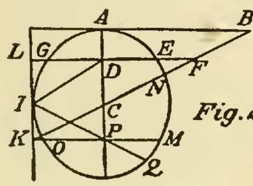

Fig. 2

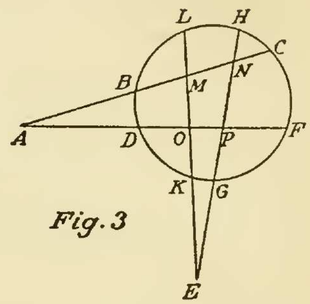

Fig. 3

MV, *SK*; and if through two of the four points *A*, *K*, μ, *V*, which can not lie in the same line with points *M*, *S*, and also through points *K*, *V*, a circle passes cutting the lines *MV*, *MP*, *SV*, *SK* at points *O*, *P*, *Q*, *N*, then I say that the lines *MS*, *NO*, *PQ*, are of the same order.

Lemma II

If through the same line several planes are passed, and are cut by another plane, all lines of intersection of these planes are of the same order as the line through which these planes pass.

On the basis of these two lemmas and several easy deductions from them, we can demonstrate that if the same things are granted as for the first lemma, that is, through points *K*, *V*, any conic section whatever passes cutting the lines *MK*, *MV*, *SK*, *SV* in

[1] [In this translation, the word "line," meaning a straight line-segment, will be used for "droite."]

points P, O, N, Q, then the lines MS, NO, PQ will be of the same order. This constitutes a third lemma.[1]

By means of these three lemmas and certain deductions, therefrom, we propose to derive a complete ordered sequence of conics,[2] that is to say, all the properties of diameters and other straight lines,[3] of tangents, &c., the construction of the cone from substantially these data, the description of conic sections by points, etc.

Having done this, we shall state the properties which follow, doing this in a more general manner than usual. Take for example, the following: If in the plane MSQ, in the conic PKV, there are drawn the lines AK, AV, cutting the conic in points P, K, Q, V; and if from two of these four points, which do not lie in the same line with point A,—say the points K, V, and through two points N, O, taken on the conic, there are produced four lines KN, KO, VN, VO, cutting the lines AV, AP at points L, M, T, S,—then I maintain that the proportion composed of the ratios of the line PM to the line MA, and of the line AS to the line SQ, is the same as the proportion composed of the ratio of the line PL to the line LA, and of the line AT to the line TQ.

We can also demonstrate that if there are three lines DE, DG, DH that are cut by the lines AP, AR at points F, G, H, C, γ, B, and if the point E be fixed in the line DC, the proportion composed of the ratios of the rectangle $EF.FG$ to the rectangle $EC.C\gamma$, and of the line $A\gamma$ to the line AG, is the same as the ratio of the rectangle $EF.FH$ to the rectangle $EC.CB$, and of the line AB to the line AH. The same is also true with respect to the ratio of the rectangle $FE.FD$ to the rectangle $CE.CD$. Consequently, if a conic section passes through the points E, D, cutting the lines AH, AB in points P, K, R, ψ, the proportion composed of the ratio of the rectangle of these lines EF, FC, to the rectangle of the lines EC, $C\gamma$, and of the line γA to the line AG, will be the same as the ratio of the rectangle of the lines FK, FP, to the rectangle of the lines CR, $C\psi$, and of the rectangle of the lines AR, $A\psi$, to the rectangle of the lines AK, AP.

We can also show that if four lines AC, AF, EH, EL intersect in points N, P, M, O, and if a conic section cuts these lines in

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[1] [This involves the so-called "Mystic Hexagram," the dual of Brianchon's Theorem given on page 331. Pascal did not state the hexagram theorem in the form commonly seen in textbooks.]

[2] . . . des éléments coniques complets.

[3] . . . et côtés droits.

points *C, B, F, D, H, G, L, K,* the proportion consisting of the ratios of rectangle *MC.MB* to rectangle *PF.PD,* and of rectangle *AD.AF* to rectangle *AB.AC,* is the same as the proportion composed of the ratios of rectangle *ML.MK* to the rectangle *PH.PG,* and of rectangle *EH.EG* to rectangle *EK.EL.*

We can also demonstrate a property stated below, due to *M.* Desargues of Lyons, one of the great geniuses of this time and well versed in mathematics, particularly in conics, whose writings on this subject although few in number give abundant proof of his knowledge to those who seek for information. I should like to say that I owe the little that I have found on this subject to his writings, and that I have tried to imitate his method, as far as possible, in which he has treated the subject without making use of the triangle through the axis.

Giving a general treatment of conic sections, the following is the remarkable property under discussion: If in the plane *MSQ* there is a conic section *PQN,* on which are taken four points *K, N, O, V* from which are drawn the lines *KN, KO, VN, VO,* in such a way that through the same four points only two lines may pass, and if another line cuts the conic at points *R,* ψ, and the lines *KN, KO, VN, VO,* in points *X, Y, Z,* δ, then as the rectangle $ZR.Z\psi$ is to the rectangle $\gamma R.\gamma\psi$, so the rectangle $\delta R.\delta\psi$ is to the rectangle $XR.X\psi$.

We can also prove that, if in the plane of the hyperbola, the ellipse, or the circle *AGE* of which the center is *C,* the line *AB* is drawn touching the section at *A,* and if having drawn the diameter we take line *AB* such that its square shall be equal to the square of the figure,[1] and if *CB* is drawn, then any line such as *DE,* parallel to line *AB* and cutting the section in *E,* and the lines *AC, CB* in points *D, F,* then if the section *AGE* is an ellipse or a circle, the sum of the squares of the lines *DE, DF* will be equal to the square of the line *AB*; and in the hyperbola, the difference between the same squares of the lines *DE, DF* will be equal to the square of the line *AB.*

[1] [In order that the square of segment *AB,* which is equal to $DE + DF$, shall be equal to one fourth of the circumscribed rectangle, the conic must be a circle. If the conic is an ellipse, *AB* will be taken equal to the axis which is perpendicular to *CA.*

Desargues treated analogous questions in his *Brouillon Projet* (Œuvres de Desargues, I, p. 202 et p. 284).]

Attention should be called to the fact that the statements of both Desargues and Pascal immediately lead up to the equation of conics.

We can also deduce [from this] several problems; for example:

From a given point to draw a tangent to a given conic section.

To find two diameters that meet in a given angle.

To find two diameters cutting at a given angle and having a given ratio.

There are many other problems and theorems, and many deductions which can be made from what has been stated above, but the distrust which I have, due to my little experience and capacity, does not allow me to go further into the subject until it has passed the examination of able men who may be willing to take this trouble. After that if someone thinks the subject worth continuing, I shall endeavor to extend it as far as God gives me the strength.

At Paris, M.DC.XL.