Pierre Simon Laplace (1749-1827)
& Adrien-Marie Legendre (1752-1833)

Laplace: - mainly an applied mathematician / mathematical physicist
- adapted to each change of régime in France from the Revolution through the Bourbon Restoration (kept a low profile, especially in times of uncertainty)
- His major work was his Mécanique Celeste (5 volumes, 1799-1825), summarized and extended previous work in Newtonian mechanics and mathematical astronomy.
  - added to the theory of tides
  - stability of the solar systems

- Two major bits of applied math came out of this:

Laplace's Equation: If φ is the potential energy of a particle in a conservative vector (force) field, then

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=\nabla\cdot\nabla\varphi=\nabla^2\varphi=\Delta\varphi \leftarrow \text{"Laplacian" operator}$$

Laplace transform: If $f(t)$ is a function defined & integrable for $t \geq 0$, then the Laplace transform of f is $F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

This allows one in many applications to change a problem to one involving better-behaved functions.

Legendre: - born into a wealthy family, but lost his personal fortune in 1793 when the Revolution got really radical

- studied math & physics, initially just for fun.
- contributed to several areas of mathematics:
  algebra, number theory, analysis, statistics, geometry, and various parts of applied mathematics

~~Laplace &~~ Legendre transform: (applications in mechanics & thermodynamics)

Let $p = \frac{\partial f}{\partial x}$. The Legendre transform of $f(x)$ is then

$$f^*(p) = \sup\{ px - f(x) \mid x \in \mathrm{dom}(f) \}$$

where p is held constant.

- In statistics & related areas, he contributed the method of least squares. (1806)
- In geometry, he showed that Euclid's Parallel Postulate (Post. V) is equivalent to the existence of a single square.
- In number theory, he
  - proved Fermat's Last Theorem for $n=5$
  - conjectured the "Prime Number Theorem" in 1796

    If $\pi(x) = \#\text{ primes} \le x$, then $\pi(x) \sim \frac{x}{\ln(x)}$.

    (Not proved until 1898 ...)
  - conjectured the Law of Quadratic Reciprocity (previously conjectured by Euler) If $p \nmid n$, then $\underbrace{\left(\frac{n}{p}\right)}_{\text{Legendre symbol}} = \begin{cases} 1 & \text{if } n \equiv a^2 \pmod{p} \\ -1 & \text{if } n \not\equiv a^2 \pmod{p} \end{cases}$.

    Then if p & q are distinct odd primes, then

    $$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

    Legendre proved some special cases of this, but it took Gauss to fully prove it.

- In algebra, among other things he devised the Legendre polynomials, which occur in solutions to the Laplace equation, various power series expansions, trigonometric formulas, and so on.

The Legendre polynomials are a system of polynomials $P_n(x)$ [of degree $n \geq 0$] satisfying $P_n(1) = 1$

and $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$ whenever $n \neq m$.

- The fact that $P_0(x) = a$ is constant & $P_0(1) = 1$ means that

$$P_0(x) = 1.$$

- For $P_1(x)$ we have $P_1(x) = ax+b$ & $P_0(1) = 1 = a \cdot 1 + b = a+b$

$$\& \; 0 = \int_{-1}^{1} P_0(x) P_1(x)\, dx = \int_{-1}^{1} (ax+b) \cdot 1\, dx = \frac{a}{2}x^2 + bx \Big|_{-1}^{1}$$

$$= \left(\frac{a}{2} + b\right) - \left(\frac{a}{2} - b\right) = 2b \quad \Rightarrow b = 0 \Rightarrow a = 1$$

so $P_1(x) = x$.

- For $P_2(x)$ we have $P_2(x) = ax^2+bx+c$

$$1 = P_1(1) = a+b+c$$

$$0 = \int_{-1}^{1} P_2(x)P_0(x)\,dx = \int_{-1}^{1}(ax^2+bx+c)\cdot 1\,dx = a\frac{x^3}{3}+\frac{b}{2}x^2+cx\Big|_{-1}^{1}$$

$$= \left(\frac{a}{3}+\frac{b}{2}+c\right)-\left(-\frac{a}{3}+\frac{b}{2}-c\right) = \frac{2}{3}a+2c$$

$$0 = \int_{-1}^{1} P_2(x)P_1(x)\,dx = \int_{-1}^{1}(ax^2+bx+c)\cdot x\,dx = \int_{-1}^{1}(ax^3+bx^2+cx)\,dx$$

$$= a\frac{x^4}{4}+b\frac{x^3}{3}+c\frac{x^2}{2}\Big|_{-1}^{1} = \left(\frac{a}{4}+\frac{b}{3}+\frac{c}{2}\right)-\left(\frac{a}{4}-\frac{b}{3}+\frac{c}{2}\right) = \frac{2}{3}b$$

$\Rightarrow b=0$ so $a+c=1$ & $\frac{2}{3}a+2c=0=\frac{a}{3}+c$

$\Rightarrow \frac{2}{3}a+0c = 1-0 = 1 \Rightarrow a=\frac{3}{2}$ & $c=-\frac{1}{2}$

$$\therefore\ P_2(x) = \frac{3}{2}x^2-\frac{1}{2} = \frac{1}{2}(3x^2-1)$$

& so on...

Two places that Legendre polynomials happen:

1° (As introduced by Legendre in 1782) as coefficients in the expansion of Newtonian potential.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos(\gamma)}} = \sum_{n=0}^{\infty} P_n(\cos(\gamma)) \cdot \frac{(r')^n}{r^{n+1}}$$

where $\vec{x}, \vec{x}'$ are vectors; r, r' are their lengths; γ is the angle between them.

2° Trig identities: $$\frac{\sin((n+1)\theta)}{\sin(n\theta)} = \sum_{i=0}^{n} P_i(\cos(\theta)) \cdot P_{n-i}(\cos(\theta))$$