2020-11-16

Three non-Bernoullis, with a bit of Jacob Bernoulli.

Brook Taylor (1685-1731)

- studied and was later a professor at Cambridge
- admirer of Newton's and sat on the committee that Newton put together to adjudicate the priority dispute with Leibniz.
- discovered Taylor series [published in 1715]

  The Taylor series of $f(x)$ at $x=a$ is

  $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

  A natural extension of expanding functions as power series to do calculus with them, (Newton's Binomial Formula), but it's importance was not generally recognized untill 1772 by Lagrange.
- invented the calculus of finite differences [also published in 1715]

George Berkeley (1685-1753)

- primarily a philosopher and a theologian as a scholar
- also an Anglican bishop in Ireland
- as a philosopher he studied (among other things) the philosophy of science and mathematics.
- published a critizism of the foundations of calculus in 1734 called The Analyst: A Discourse Addressed to an Infidel Mathematician in which he called out mathematizians for not having a rigorous development of "fluxions" & "infinitesimals".
- various attempts were made to address his critizisms, but with at best partial success until the mid-1800s [Cauchy's work on convergence in the 1830s & Dedekind's ϵ-δ def'n of limits.]

From *The Analyst* (1734), by George Berkeley:

It must, indeed, be acknowledged, that [Newton] used Fluxions, like the Scaffold of a building, as things to be laid aside or got rid of, as soon as finite Lines were found proportional to them. But then these finite Exponents are found by the help of Fluxions. Whatever therefore is got by such Exponents and Proportions is to be ascribed to Fluxions: which must therefore be previously understood. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?

From his *Philosphical Commentaries* (1707-1708):

#354. Axiom. No reasoning about things whereof we have no idea. Therefore no reasoning about Infinitesimals.

#767. Take away the signs from Arithmetic & Algebra, & pray what remains?

Two limericks about George Berkeley’s philosophy and theology:

There was a young man who said “God
Must find it exceedingly odd
    To think that the tree
    Should continue to be
When there’s no one about in the quad.”

“Dear Sir: Your astonishment’s odd;
I *am* always about in the quad.
    And that’s why the tree
    Will continue to be
Since observed by, Yours faithfully, God.”

Colin MacLaurin (1698-1746)

- prodigy
  - entered the Univ. of Glasgow ~~at~~ at 11
  - graduated with an M.A. at 14 (with a thesis on gravity)
  - elected a professor at Univ. of Aberdeen at 19
  - became a professor at Univ. of Edinburgh in 1725 [with a recommendation from Newton]

- as a mathematician & physicist
  - did work on gravitational attraction for ellipsoids and oblate spheroids
  - calculus: he used Taylor series to do calculus problems (eg max/min)
  - tried to make the foundations of calculus rigorous in his <u>Treatise of Fluxion</u> (1742) [and the posthumous <u>Treatise of Algebra, 1748</u>]
  - attempted to treat fluxions, <u>etc</u>, axiomatically.

- He and Euler independently discovered the Euler-MacLaurin formula c. 1735

To describe the formula we first need a little bit on the Bernoulli numbers discovered by Jacob Bernoulli. (Ars Conjectandi, 1713)

These are a sequence of rational numbers,

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{-1}{30}, \quad B_5 = 0, \ \ldots$$

[All odd-numbered Bernoulli numbers after B_1 are 0.]

Several ways to define the sequence, so here's one.

$$B_n = \sum_{k=0}^{n} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{(m+1)^n}{k+1}.$$

This sequence has a lot of applications,

eg
$$\sum_{k=1}^{n} k^m = 1^m + 2^n + \cdots + n^m$$
$$= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \, n^{m+1-k} = m! \sum_{k=0}^{m} \frac{B_k \, n^{m+1-k}}{k!\,(m+1-k)!}$$

and the Taylor series for $\tan(x)$, namely

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1} \qquad \left(\text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}\right).$$

The Euler-MacLaurin formula

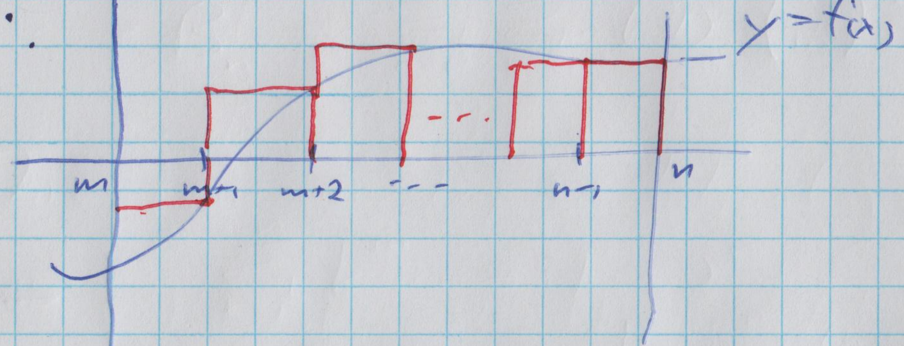
relating sums of a function evaluated at integer points to its integral on an interval

$$I = \int_m^n f(x)\,dx$$

[$f(x)$ is as differentiable as necessary on $[m,n]$.]

$$S = f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

– This is the Right-Hand Rule sum for $f(x)$ on $[m,n]$ with integer points used to define the partition.



error term – surprisingly often quite small.

$$\boxed{S - I = \sum_{k=1}^{p} \frac{B_k}{k!}\left(f^{(k-1)}(n) - f^{(k-1)}(m)\right) + R_p}.$$