2020-10-14

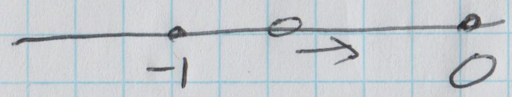
Logarithms

[Protestant reformation began in 1517 with Martin Luther posting his "95 theses". Peace of Augsburg 1555 divided up Germany into Protestant & Catholic bits]

John Napier (1550-1617)

- Scottish aristocrat (Baron of Murchiston) and a Protestant militant
- invented logarithms *Mirifici Logarithmorum Canonis Constructio* (published in 1614)
  - he constructed tables of logarithms, specifically for $10^7 \log_{1/e}\left(\frac{x}{10^7}\right)$

Basic idea: Suppose a particle is approaching the origin from the left at a speed proportional to its distance from the origin, with the particle being at $x(0)=-1$ at $t=0$.



If $x(t)$ is the position of the particle [so it's distance from 0 is $|x(t)|$] at time t and the constant of proportionality is 1, this gives us the differential equation $\frac{dx}{dt} = |x(t)| = -x(t)$ with initial condition $x(0)=-1$.

②

⌈ So $\frac{dx}{dt} = -x \Rightarrow \frac{dx}{x} = -t \Rightarrow \int \frac{dx}{x} = \int (-1)\,dt$

$\Rightarrow \ln(x) = \log_e(x) = -t$

$\Rightarrow x(t) = Ke^{-t}$ for some constant K

Find K by plugging in $x(0) = -1$, $x(0) = -1 = Ke^{-0} = K \cdot 1$

so $K = 1$, and $x(t) = -e^{-t}$, so $\left(\frac{1}{e}\right)^t = -x(t)$,

ie $t = \log_{1/e}(\underbrace{|x(t)|}_{-x(t)})$ ⌋

How did Napier compute his tables?

Used three basic tricks:

1. When z is small, then $\log_{1/e}(1-z) \approx z$.

2. Consider $s = 1 - 10^{-m} = 1 - \frac{1}{10^m}$ eg $s = 1 - 10^{-5}$

$= \frac{10^5 - 1}{10^5}$

so $\log_{1/e}(s) = \log_{1/e}(1 - 10^{-5})$

$\approx 10^{-5} = 0.00001$

$= \frac{100{,}000 - 1}{100{,}000}$

$= \frac{99{,}999}{100{,}000}$

$= 0.99999$

Continuing, consider $s^2 = s(1-10^{-5})$

$$= s - \frac{s}{10^5} = 1-10^{-5} - \frac{1-10^{-5}}{10^5}$$

& so on   eg $s^3 = s^2 - \frac{s^2}{10^5}$

3° For values of $z \approx 0$, $\log_{ue}(1-z)$ is smooth and pretty close to linear, so linear interpolation gives good results.

It still took him years, and the base was awkward, but he constructed table accurate to 5 decimal places or more.

Why do log tables matter? You can turn complex calculations into simpler ones.

eg $x \cdot y \Leftrightarrow \log_a(x \cdot y) = \log_a(x) + \log_a(y)$

$$\Rightarrow x \cdot y = a^{\log_a(x) + \log_a(y)}$$

If the base (eg $a = \frac{1}{e}$) is difficult to work with, this limits the utility.

Henry Briggs (1561-1631)

- worked with Napier from 1615 on to create log tables for $\log_{10}$ (for which the computation of $10^{\log_{10}(a)}$ is relatively easy)

which were published in 1624.

Once $\log_{10}$ tables were available, lots of scientists (esp. astronomers) adopted them to speed up computations.
(Kepler)

Notation: $\log$ was adopted by Leibniz in 1675
and the definition that $\ln(y) = \log_e(y) = \int_1^y \frac{1}{t}\,dt$
came a year later.

Next time: astronomers!