QUADRATURE OF THE PARABOLA.

" ARCHIMEDES to Dositheus greeting.

" When I heard that Conon, who was my friend in his life-
time, was dead, but that you were acquainted with Conon and
withal versed in geometry, while I grieved for the loss not only
of a friend but of an admirable mathematician, I set myself the
task of communicating to you, as I had intended to send to
Conon, a certain geometrical theorem which had not been
investigated before but has now been investigated by me, and
which I first discovered by means of mechanics and then
exhibited by means of geometry. Now some of the earlier
geometers tried to prove it possible to find a rectilineal area
equal to a given circle and a given segment of a circle; and
after that they endeavoured to square the area bounded by the
section of the whole cone * and a straight line, assuming lemmas
not easily conceded, so that it was recognised by most people
that the problem was not solved. But I am not aware that
any one of my predecessors has attempted to square the
segment bounded by a straight line and a section of a right-
angled cone [a parabola], of which problem I have now dis-
covered the solution. For it is here shown that every segment
bounded by a.straight line and a section of a right-angled cone
[a parabola] is four-thirds of the triangle which has the same base
and equal height with the segment, and for the demonstration

* There appears to be some corruption here : the expression in the text is
τᾶς ὅλου τοῦ κώνου τομᾶς, and it is not easy to give a natural and intelligible
meaning to it. The section of ' the whole cone ' might perhaps mean a section
cutting right through it, i.e. an ellipse, and the ' straight line' might be an axis
or a diameter. But Heiberg objects to the suggestion to read τᾶς ὀξυγωνίου
κώνου τομᾶς, in view of the addition of καὶ εὐθείας, on the ground that the former
expression always signifies the whole of an ellipse, never a segment of it
(*Quaestiones Archimedeae*, p. 149).

of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted* no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof (στοιχεῖα κωνικὰ χρείαν ἔχοντα ἐς τὰν ἀπόδειξιν). Farewell."

Proposition 1.

If from a point on a para-bola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if QQ' be a chord parallel to the tangent to the parabola at P and meeting PV in V, then

$$QV = VQ'.$$

Conversely, *if QV = VQ', the chord QQ' will be parallel to the tangent at P.*

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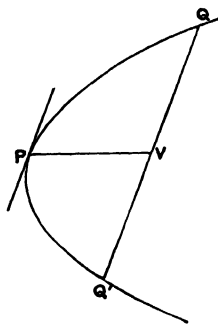
* The Greek of this passage is: συμβαίνει δὲ τῶν προειρημένων θεωρημάτων ἕκαστον μηδὲν ἧσσον τῶν ἄνευ τούτου τοῦ λήμματος ἀποδεδειγμένων πεπιστευκέναι. Here it would seem that πεπιστευκέναι must be wrong and that the passive should have been used.

Proposition 2.

If in a parabola QQ' be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets QQ' in V and the tangent at Q to the parabola in T, then

$$PV = PT.$$
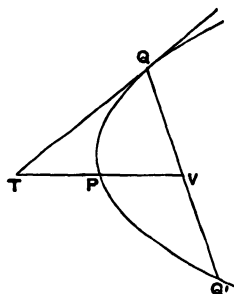
Proposition 3.

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V' respectively, then

$$PV : PV' = QV^2 : Q'V'^2.$$

"*And these propositions are proved in the elements of conics.**"

Proposition 4.

If Qq be the base of any segment of a parabola, and P the vertex of the segment, and if the diameter through any other point R meet Qq in O and QP (produced if necessary) in F, then
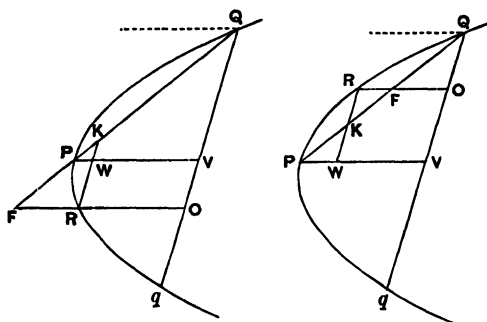
$$QV : VO = OF : FR.$$

Draw the ordinate RW to PV, meeting QP in K.

* i.e. in the treatises on conics by Euclid and Aristaeus.

Then $PV : PW = QV^2 : RW^2$;

whence, by parallels,

$$PQ : PK = PQ^2 : PF^2.$$

In other words, PQ, PF, PK are in continued proportion; therefore

$$PQ : PF = PF : PK$$
$$= PQ \pm PF : PF \pm PK$$
$$= QF : KF.$$

Hence, by parallels,

$$QV : VO = OF : FR.$$

[It is easily seen that this equation is equivalent to a change of axes of coordinates from the tangent and diameter to new axes consisting of the chord Qq (as axis of x, say) and the diameter through Q (as axis of y).

For, if $QV = a$, $PV = \dfrac{a^2}{p}$, where p is the parameter of the ordinates to PV.

Thus, if $QO = x$, and $RO = y$, the above result gives

$$\frac{a}{x - a} = \frac{OF}{OF - y},$$

whence $\dfrac{a}{2a - x} = \dfrac{OF}{y} = \dfrac{x \cdot \dfrac{a}{p}}{y},$
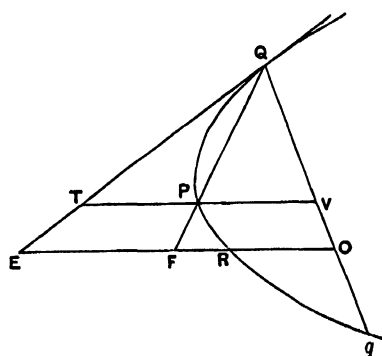
or $py = x(2a - x).$]

Proposition 5.

If Qq be the base of any segment of a parabola, P the vertex of the segment, and PV its diameter, and if the diameter of the parabola through any other point R meet Qq in O and the tangent at Q in E, then

$$QO : Oq = ER : RO.$$

Let the diameter through R meet QP in F.

Then, by Prop. 4,
$$QV : VO = OF : FR.$$

Since $QV = Vq$, it follows that

$$QV : qO = OF : OR \dots\dots\dots\dots(1).$$

Also, if VP meet the tangent in T,

$$PT = PV, \text{ and therefore } EF = OF.$$

Accordingly, doubling the antecedents in (1), we have

$$Qq : qO = OE : OR,$$

whence $\qquad QO : Oq = ER : RO.$

Propositions 6, 7*.

Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD in which the angle C is right or obtuse be suspended from B and O, so that C is attached to O and CD is in the same vertical line with O. Then, if P be such an area as, when suspended from A, will keep the system in equilibrium,

$$P = \tfrac{1}{3} \triangle BCD.$$

Take a point E on OB such that $BE = 2OE$, and draw EFH parallel to OCD meeting BC, BD in F, H respectively. Let G be the middle point of FH.
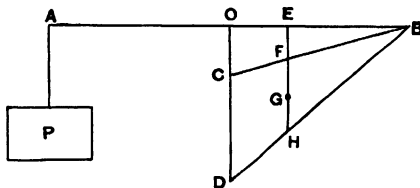
Then G is the centre of gravity of the triangle BCD.

Hence, if the angular points B, C be set free and the triangle be suspended by attaching F to E, the triangle will hang in the same position as before, because EFG is a vertical straight line. "For this is proved†."

Therefore, as before, there will be equilibrium.

Thus $P : \triangle BCD = OE : AO$

                                    $= 1 : 3,$

or $P = \tfrac{1}{3} \triangle BCD.$

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* In Prop. 6 Archimedes takes the separate case in which the angle BCD of the triangle is a right angle so that C coincides with O in the figure and F with E. He then proves, in Prop. 7, the same property for the triangle in which BCD is an obtuse angle, by treating the triangle as the difference between two right-angled triangles BOD, BOC and using the result of Prop. 6. I have combined the two propositions in one proof, for the sake of brevity. The same remark applies to the propositions following Props. 6, 7.

† Doubtless in the lost book περὶ ζυγῶν. Cf. the Introduction, Chapter II., *ad fin.*

Propositions 8, 9.

Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD, right-angled or obtuse-angled at C, be suspended from the points B, E on OB, the angular point C being so attached to E that the side CD is in the same vertical line with E. Let Q be an area such that

$$AO : OE = \triangle BCD : Q.$$

Then, if an area P suspended from A keep the system in equilibrium,

$$P < \triangle BCD \; but \; > Q.$$

Take G the centre of gravity of the triangle BCD, and draw GH parallel to DC, i.e. vertically, meeting BO in H.
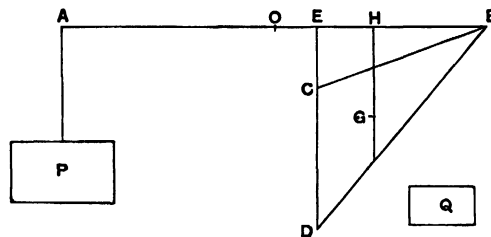
We may now suppose the triangle BCD suspended from H, and, since there is equilibrium,

$$\triangle BCD : P = AO : OH \dots\dots\dots\dots\dots(1),$$

whence $\qquad\qquad P < \triangle BCD.$

Also $\qquad\qquad \triangle BCD : Q = AO : OE.$

Therefore, by (1), $\quad \triangle BCD : Q > \triangle BCD : P,$

and $\qquad\qquad\qquad P > Q.$

Propositions 10, 11.

Suppose a lever AOB placed horizontally and supported at O, its middle point. Let CDEF be a trapezium which can be so placed that its parallel sides CD, FE are vertical, while C is vertically below O, and the other sides CF, DE meet in B. Let EF meet BO in H, and let the trapezium be suspended by attaching F to H and C to O. Further, suppose Q to be an area such that
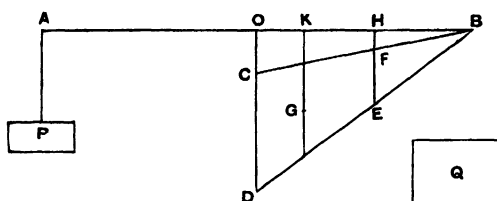
$$AO : OH = (trapezium \; CDEF) : Q.$$

Then, if P be the area which, when suspended from A, keeps the system in equilibrium,

$$P < Q.$$

The same is true in the particular case where the angles at C, F are right, and consequently C, F coincide with O, H respectively.

Divide OH in K so that

$$(2CD + FE) : (2FE + CD) = HK : KO.$$

Draw KG parallel to OD, and let G be the middle point of the portion of KG intercepted within the trapezium. Then G is the centre of gravity of the trapezium [*On the equilibrium of planes,* I. 15].

Thus we may suppose the trapezium suspended from K, and the equilibrium will remain undisturbed.

Therefore

$$AO : OK = (\text{trapezium } CDEF) : P,$$

and, by hypothesis,

$$AO : OH = (\text{trapezium } CDEF) : Q.$$

Since $OK < OH$, it follows that

$$P < Q.$$

Propositions 12, 13.

If the trapezium CDEF be placed as in the last propositions, except that CD is vertically below a point L on OB instead of being below O, and the trapezium is suspended from L, H, suppose that Q, R are areas such that
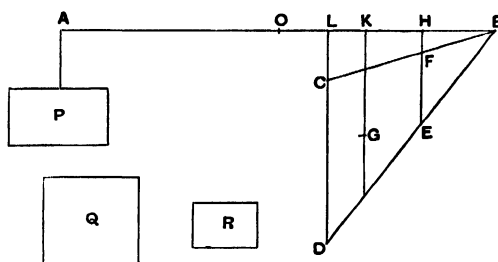
$$AO : OH = (\text{trapezium } CDEF) : Q,$$

and $\qquad AO : OL = (\text{trapezium } CDEF) : R.$

If then an area P suspended from A keep the system in equilibrium,

$$P > R \text{ but } < Q.$$

Take the centre of gravity G of the trapezium, as in the last propositions, and let the line through G parallel to DC meet OB in K.

Then we may suppose the trapezium suspended from K, and there will still be equilibrium.

Therefore (trapezium $CDEF$) : $P = AO : OK$.

Hence

$$(\text{trapezium } CDEF) : P > (\text{trapezium } CDEF) : Q,$$

but $< (\text{trapezium } CDEF) : R.$

It follows that $P < Q$ but $> R.$

Propositions 14, 15.

Let Qq be the base of any segment of a parabola. Then, if two lines be drawn from Q, q, each parallel to the axis of the parabola and on the same side of Qq as the segment is, either (1) the angles so formed at Q, q are both right angles, or (2) one is acute and the other obtuse. In the latter case let the angle at q be the obtuse angle.

Divide Qq into any number of equal parts at the points $O_1, O_2, \ldots O_n$. Draw through q, $O_1, O_2, \ldots O_n$ diameters of the parabola meeting the tangent at Q in $E, E_1, E_2, \ldots E_n$ and the parabola itself in q, $R_1, R_2, \ldots R_n$. Join $QR_1, QR_2, \ldots QR_n$ meeting qE, $O_1E_1, O_2E_2, \ldots O_{n-1}E_{n-1}$ in $F, F_1, F_2, \ldots F_{n-1}$.

Let the diameters Eq, E_1O_1, ... E_nO_n meet a straight line QOA drawn through Q perpendicular to the diameters in the points O, H_1, H_2, ... H_n respectively. (In the particular case where Qq is itself perpendicular to the diameters q will coincide with O, O_1 with H_1, and so on.)

It is required to prove that

(1) $\triangle EqQ < 3(sum\ of\ trapezia\ FO_1,\ F_1O_2,\ ...\ F_{n-1}O_n\ and\ \triangle E_nO_nQ)$,

(2) $\triangle EqQ > 3(sum\ of\ trapezia\ R_1O_2, R_2O_3, ...\ R_{n-1}O_n\ and\ \triangle R_nO_nQ)$.
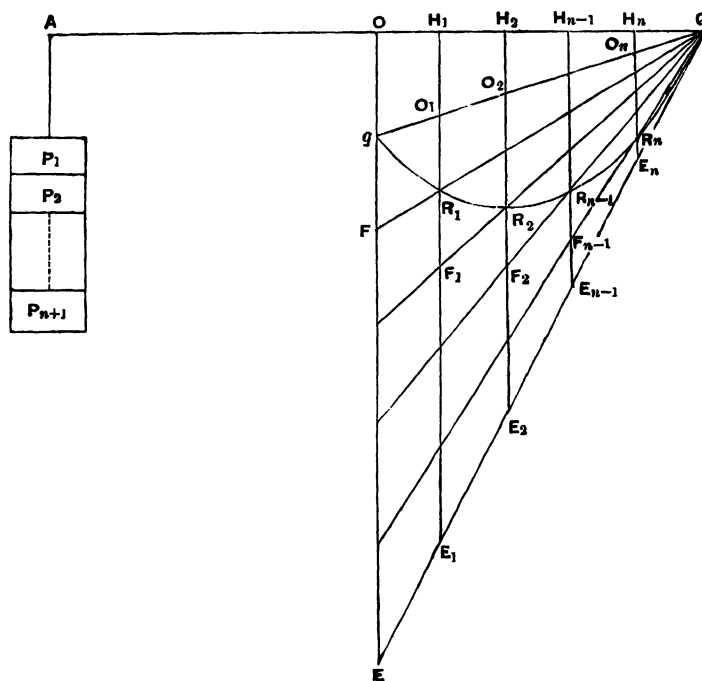
Suppose AO made equal to OQ, and conceive QOA as a lever placed horizontally and supported at O. Suppose the triangle EqQ suspended from OQ in the position drawn, and suppose that the trapezium EO_1 in the position drawn is balanced by an area P_1 suspended from A, the trapezium E_1O_2 in the position drawn is balanced by the area P_2 suspended

from A, and so on, the triangle $E_n O_n Q$ being in like manner balanced by P_{n+1}.

Then $P_1 + P_2 + \ldots + P_{n+1}$ will balance the whole triangle EqQ as drawn, and therefore

$$P_1 + P_2 + \ldots + P_{n+1} = \tfrac{1}{3} \triangle EqQ. \qquad \text{[Props. 6, 7]}$$

Again $\qquad AO : OH_1 = QO : OH_1$

$$= Qq : qO_1$$

$$= E_1 O_1 : O_1 R_1 \text{ [by means of Prop. 5]}$$

$$= (\text{trapezium } EO_1) : (\text{trapezium } FO_1);$$

whence [Props. 10, 11]

$$(FO_1) > P_1.$$

Next $\qquad AO : OH_1 = E_1 O_1 : O_1 R_1$

$$= (E_1 O_2) : (R_1 O_2) \ldots\ldots\ldots\ldots(\alpha),$$

while $\qquad AO : OH_2 = E_2 O_2 : O_2 R_2$

$$= (E_1 O_2) : (F_1 O_2) \ldots\ldots\ldots\ldots(\beta);$$

and, since (α) and (β) are simultaneously true, we have, by Props. 12, 13,

$$(F_1 O_2) > P_2 > (R_1 O_2).$$

Similarly it may be proved that

$$(F_2 O_3) > P_3 > (R_2 O_3),$$

and so on.

Lastly [Props. 8, 9]

$$\triangle E_n O_n Q > P_{n+1} > \triangle R_n O_n Q.$$

By addition, we obtain

(1) $(FO_1) + (F_1 O_2) + \ldots + (F_{n-1} O_n) + \triangle E_n O_n Q > P_1 + P_2 + \ldots + P_{n+1}$

$$> \tfrac{1}{3} \triangle EqQ,$$

or $\qquad \triangle EqQ < 3 (FO_1 + F_1 O_2 + \ldots + F_{n-1} O_n + \triangle E_n O_n Q).$

(2) $(R_1 O_2) + (R_2 O_3) + \ldots + (R_{n-1} O_n) + \triangle R_n O_n Q < P_2 + P_3 + \ldots + P_{n+1}$

$$< P_1 + P_2 + \ldots + P_{n+1}, \text{ a fortiori},$$

$$< \tfrac{1}{3} \triangle EqQ,$$

or $\qquad \triangle EqQ > 3 (R_1 O_2 + R_2 O_3 + \ldots + R_{n-1} O_n + \triangle R_n O_n Q).$

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Proposition 16.

Suppose Qq to be the base of a parabolic segment, q being not more distant than Q from the vertex of the parabola. Draw through q the straight line qE parallel to the axis of the parabola to meet the tangent at Q in E. It is required to prove that

$$\text{(area of segment)} = \tfrac{1}{3} \, \triangle EqQ.$$

For, if not, the area of the segment must be either greater or less than $\frac{1}{3} \triangle EqQ$.

I. Suppose the area of the segment greater than $\frac{1}{3} \triangle EqQ$. Then the excess can, if continually added to itself, be made to exceed $\triangle EqQ$. And it is possible to find a submultiple of the triangle EqQ less than the said excess of the segment over $\frac{1}{3} \triangle EqQ$.
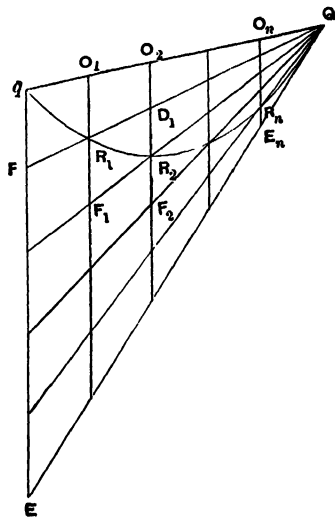
Let the triangle FqQ be such a submultiple of the triangle EqQ. Divide Eq into equal parts each equal to qF, and let all the points of division including F be joined to Q meeting the parabola in $R_1, R_2, \ldots$ R_n respectively. Through $R_1, R_2, \ldots R_n$ draw diameters of the parabola meeting qQ in $O_1, O_2, \ldots O_n$ respectively.

Let O_1R_1 meet QR_2 in F_1.

Let O_2R_2 meet QR_1 in D_1 and QR_3 in F_2.

Let O_3R_3 meet QR_2 in D_2 and QR_4 in F_3, and so on.

We have, by hypothesis,

$$\triangle FqQ < \text{(area of segment)} - \tfrac{1}{3} \triangle EqQ,$$

or (area of segment) $- \triangle FqQ > \frac{1}{3} \triangle EqQ$ .... (α).

Now, since all the parts of qE, as qF and the rest, are equal, $O_1R_1 = R_1F_1$, $O_2D_1 = D_1R_2 = R_2F_2$, and so on; therefore

$$\triangle FqQ = (FO_1 + R_1O_2 + D_1O_3 + ...)$$
$$= (FO_1 + F_1D_1 + F_2D_2 + ... + F_{n-1}D_{n-1} + \triangle E_nR_nQ)...(\beta).$$

But

(area of segment) $< (FO_1 + F_1O_2 + ... + F_{n-1}O_n + \triangle E_nO_nQ)$.

Subtracting, we have

(area of segment) $- \triangle FqQ < (R_1O_2 + R_2O_3 + ...$
$$+ R_{n-1}O_n + \triangle R_nO_nQ),$$

whence, *a fortiori*, by (α),

$$\tfrac{1}{3}\triangle EqQ < (R_1O_2 + R_2O_3 + ... + R_{n-1}O_n + \triangle R_nO_nQ).$$

But this is impossible, since [Props. 14, 15]

$$\tfrac{1}{3}\triangle EqQ > (R_1O_2 + R_2O_3 + ... + R_{n-1}O_n + \triangle R_nO_nQ).$$

Therefore

(area of segment) $\not> \tfrac{1}{3}\triangle EqQ$.

II. If possible, suppose the area of the segment less than $\tfrac{1}{3}\triangle EqQ$.

Take a submultiple of the triangle EqQ, as the triangle FqQ, less than the excess of $\tfrac{1}{3}\triangle EqQ$ over the area of the segment, and make the same construction as before.

Since $\triangle FqQ < \tfrac{1}{3}\triangle EqQ - $ (area of segment),

it follows that

$\triangle FqQ + $ (area of segment) $< \tfrac{1}{3}\triangle EqQ$
$$< (FO_1 + F_1O_2 + ... + F_{n-1}O_n + \triangle E_nO_nQ).$$
$$\text{[Props. 14, 15]}$$

Subtracting from each side the area of the segment, we have

$\triangle FqQ < $ (sum of spaces qFR_1, $R_1F_1R_2$, ... E_nR_nQ)
$$< (FO_1 + F_1D_1 + ... + F_{n-1}D_{n-1} + \triangle E_nR_nQ), \text{ a fortiori};$$

which is impossible, because, by (β) above,

$$\triangle FqQ = FO_1 + F_1D_1 + ... + F_{n-1}D_{n-1} + \triangle E_nR_nQ.$$

Hence (area of segment) $\not< \tfrac{1}{3}\triangle EqQ$.

Since then the area of the segment is neither less nor greater than $\tfrac{1}{3}\triangle EqQ$, it is equal to it.

Proposition 17.

It is now manifest that *the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.*

Let Qq be the base of the segment, P its vertex. Then PQq is the inscribed triangle with the same base as the segment and equal height.

Since P is the vertex* of the segment, the diameter through P bisects Qq. Let V be the point of bisection.

Let VP, and qE drawn parallel to it, meet the tangent at Q in T, E respectively.
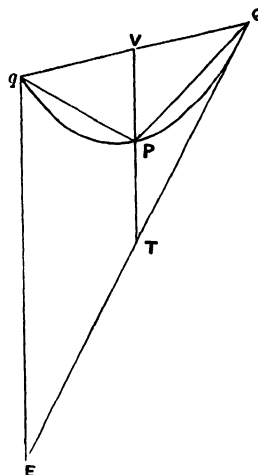
Then, by parallels,

$$qE = 2VT,$$

and $PV = PT,$ [Prop. 2]

so that $VT = 2PV.$

Hence $\triangle EqQ = 4\triangle PQq.$

But, by Prop. 16, the area of the segment is equal to $\frac{1}{3}\triangle EqQ.$

Therefore   (area of segment) $= \frac{4}{3}\triangle PQq.$

Def. "In segments bounded by a straight line and any curve I call the straight line the **base**, and the **height** the greatest perpendicular drawn from the curve to the base of the segment, and the **vertex** the point from which the greatest perpendicular is drawn."

* It is curious that Archimedes uses the terms *base* and *vertex* of a segment here, but gives the definition of them later (at the end of the proposition). Moreover he assumes the converse of the property proved in Prop. 18.

Proposition 18.

If Qq be the base of a segment of a parabola, and V the middle point of Qq, and if the diameter through V meet the curve in P, then P is the vertex of the segment.
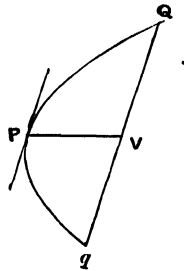

For Qq is parallel to the tangent at P [Prop. 1]. Therefore, of all the perpendiculars which can be drawn from points on the segment to the base Qq, that from P is the greatest. Hence, by the definition, P is the vertex of the segment.

Proposition 19.

If Qq be a chord of a parabola bisected in V by the diameter PV, and if RM be a diameter bisecting QV in M, and RW be the ordinate from R to PV, then

$$PV = \tfrac{4}{3}RM.$$
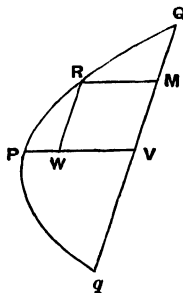
For, by the property of the parabola,

$$PV : PW = QV^2 : RW^2$$
$$= 4RW^2 : RW^2,$$

so that $\qquad\qquad PV = 4PW,$

whence $\qquad\qquad PV = \tfrac{4}{3}RM.$

Proposition 20.

If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq.

For the chord Qq is parallel to the tangent at P, and the triangle PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q.

Therefore the triangle PQq is greater than half the segment.

COR. It follows that *it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.*
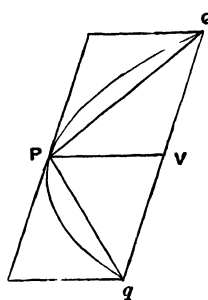
Proposition 21.

If Qq be the base, and P the vertex, of any parabolic segment, and if R be the vertex of the segment cut off by PQ, then

$$\triangle PQq = 8 \triangle PRQ.$$

The diameter through R will bisect the chord PQ, and therefore also QV, where PV is the diameter bisecting Qq. Let the diameter through R bisect PQ in Y and QV in M. Join PM.

By Prop. 19,

$$PV = \tfrac{4}{3}RM.$$
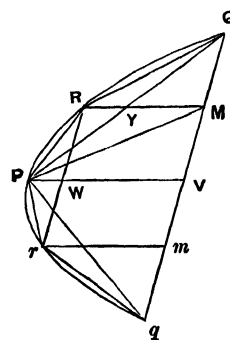
Also $PV = 2YM.$

Therefore   , $YM = 2RY,$

and $\triangle PQM = 2 \triangle PRQ.$

Hence $\triangle PQV = 4 \triangle PRQ,$

and $\triangle PQq = 8 \triangle PRQ.$

Also, if RW, the ordinate from R to PV, be produced to meet the curve again in r,

$$RW = rW,$$

and the same proof shows that

$$\triangle PQq = 8\triangle Prq.$$

Proposition 22.

If there be a series of areas A, B, C, D, ... each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then
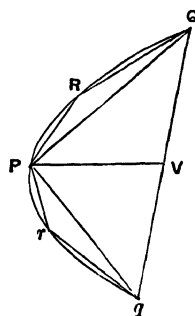
$$(A + B + C + D + ...) < (area\ of\ segment\ PQq).$$

For, since $\triangle PQq = 8\triangle PRQ = 8\triangle Pqr$, where R, r are the vertices of the segments cut off by PQ, Pq, as in the last proposition,

$$\triangle PQq = 4(\triangle PQR + \triangle Pqr).$$

Therefore, since $\triangle PQq = A$,

$$\triangle PQR + \triangle Pqr = B.$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C, and so on.

Therefore $A + B + C + D + ...$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.

Proposition 23.

Given a series of areas A, B, C, D, ... Z, of which A is the greatest, and each is equal to four times the next in order, then

$$A + B + C + ... + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

Take areas b, c, d, ... such that

$$b = \tfrac{1}{3}B,$$

$$c = \tfrac{1}{3}C,$$

$$d = \tfrac{1}{3}D, \text{ and so on.}$$

Then, since $\qquad b = \tfrac{1}{3}B,$

and $\qquad\qquad B = \tfrac{1}{4}A,$

$$B + b = \tfrac{1}{3}A.$$

Similarly $\qquad C + c = \tfrac{1}{3}B.$

....

Therefore

$$B + C + D + ... + Z + b + c + d + ... + z = \tfrac{1}{3}(A + B + C + ... + Y).$$

But $\qquad b + c + d + ... + y = \tfrac{1}{3}(B + C + D + ... + Y).$
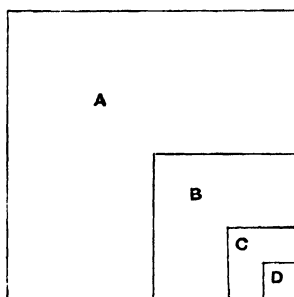
Therefore, by subtraction,

$$B + C + D + ... + Z + z = \tfrac{1}{3}A$$

or $\qquad\qquad A + B + C + ... + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$

$\Big[$ The algebraical equivalent of this result is of course

$$1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + \ldots + (\tfrac{1}{4})^{n-1} = \tfrac{4}{3} - \tfrac{1}{3}(\tfrac{1}{4})^{n-1}$$

$$= \frac{1 - (\tfrac{1}{4})^n}{1 - \tfrac{1}{4}}. \Big]$$

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $\qquad K = \tfrac{4}{3} \triangle PQq,$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K.

For, if the segment be not equal to K, it must either be greater or less.

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K.
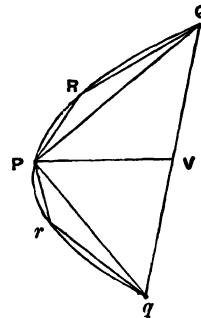
Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A + B + C + \ldots + Z < \tfrac{4}{3}A,$$

where $\qquad A = \triangle PQq.$

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\triangle PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A + B + C + \ldots + X + \tfrac{1}{3}X = \tfrac{4}{3}A \qquad \text{[Prop. 23]}$$
$$= K.$$

Now, since K exceeds $A + B + C + \ldots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \ldots + X > (\text{the segment});$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$(\text{area of segment } PQq) = K = \tfrac{4}{3}\triangle PQq.$$