Mathematics 3810H – Ancient and classical mathematics
Trent University, Fall 2017

Magnitudes and Proportions

Eudoxus of Cnidus (*c.* 410–350 B.C.) was one of the great early Greek mathematicians, who also worked in astronomy and philosophy. Unfortunately, all of his writings are now lost, and we only have the word of (sometimes very much) later commentators for what he accomplished. The theory of proportions Eudoxus is supposed to have developed allowed ancient Greek mathematicians to rigourously handle quantities that were not necessarily rational. Some of it seems very strange to modern eyes and ears, though. For one thing, the quantities in question were conceived of as geometric magnitudes, such as length or area, rather than numbers in a number system extending the rationals.

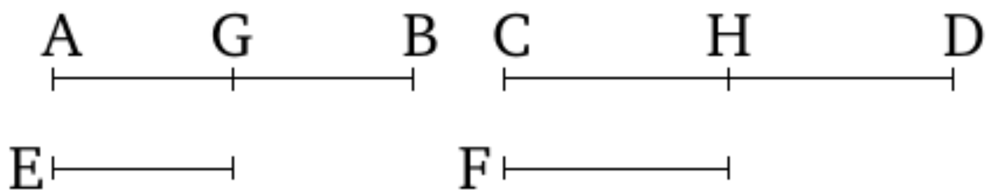
The definitions and proposition below are taken from Book V of Euclid's *Elements*, which is believed to be based on Eudoxus' work on proportions.*

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.
2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.
3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.
4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.
5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.
6. And let magnitudes having the same ratio be called proportional.
7. And when for equal multiples (as in Definition 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, then the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.
8. And a proportion in three terms is the smallest (possible).
9. And when three magnitudes are proportional, the first is said to have a squared ratio to the third with respect to the second.
10. And when four magnitudes are (continuously) proportional, the first is said to have a cubed ratio to the fourth with respect to the second. And so on, similarly, in successive order, whatever the (continuous) proportion might be.
11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.
12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following.
13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.

* Taken from Richard Fitzpatrick's *Euclid's Elements in Greek*, which gives the Greek text in parallel with an English translation. Words and phrases in parentheses are interpolations by the translator to make the otherwise pretty direct English translation clearer.

14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the same following (magnitude).
15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the same following (magnitude).
16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following.
17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (or ex aequali) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes).
18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (*i.e.* the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes.

Proposition 1. If there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second).



Let there be any number of magnitudes whatsoever, AB, CD, (which are) equal multiples, respectively, of some (other) magnitudes, E, F, of equal number (to them). I say that as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

For since AB, CD are equal multiples of E, F, thus as many magnitudes as (there) are in AB equal to E, so many (are there) also in CD equal to F . Let AB have been divided into magnitudes AG, GB, equal to E, and CD into (magnitudes) CH, HD, equal to F . So, the number of (divisions) AG, GB will be equal to the number of (divisions) CH, HD. And since AG is equal to E, and CH to F, AG (is) thus equal to E, and AG, CH to E, F. So, for the same (reasons), GB is equal to E, and GB, HD to E, F. Thus, as many (magnitudes) as (there) are in AB equal to E, so many (are there) also in AB, CD equal to E, F. Thus, as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show.