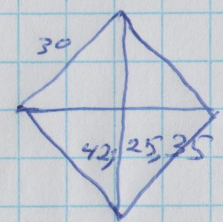
2022-01-29

(A bit of) Mesopotamian Geometry and (an example of) Algebra

Ancient Mesopotamian geometry was at least as sophisticated as that of their Egyptian contemporaries.

- we know they were aware of at least some instances of the Pythagorean Theorem

  eg the diagram on YBC 2789



$2 \times 30^2 = 42;25,35$

and a tablet from the Seleucid era (c. 300 B.C.) has a problem with answer:

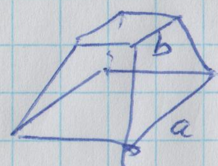
A reed stands against a wall and the top slides down 3 units while the bottom slides away 9 units. How long is the reed?
(Answer: 15 units).

This need both the Pythagorean Theorem and some algebra

- we know they computed areas & volumes of various shapes both approximately & exactly.

  - computation from various tablets involving circular areas suggest $\pi \approx 3$, or $\pi \approx 3\frac{1}{8}$, or $\pi \approx \frac{31}{10}$, or $\pi \approx \sqrt{10}$, ...

  - they also computed, like the Egyptians the volumes of truncated pyramids



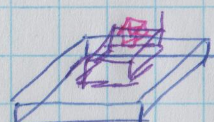
$\updownarrow h$ Approximate value $V = \left(\frac{a+b}{2}\right)^2 h$

Also used the exact value,

$$V = \left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right] h$$

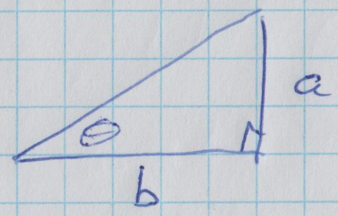
which is actually exact.

  - computed the volume of a ziggurat (step-pyramid)



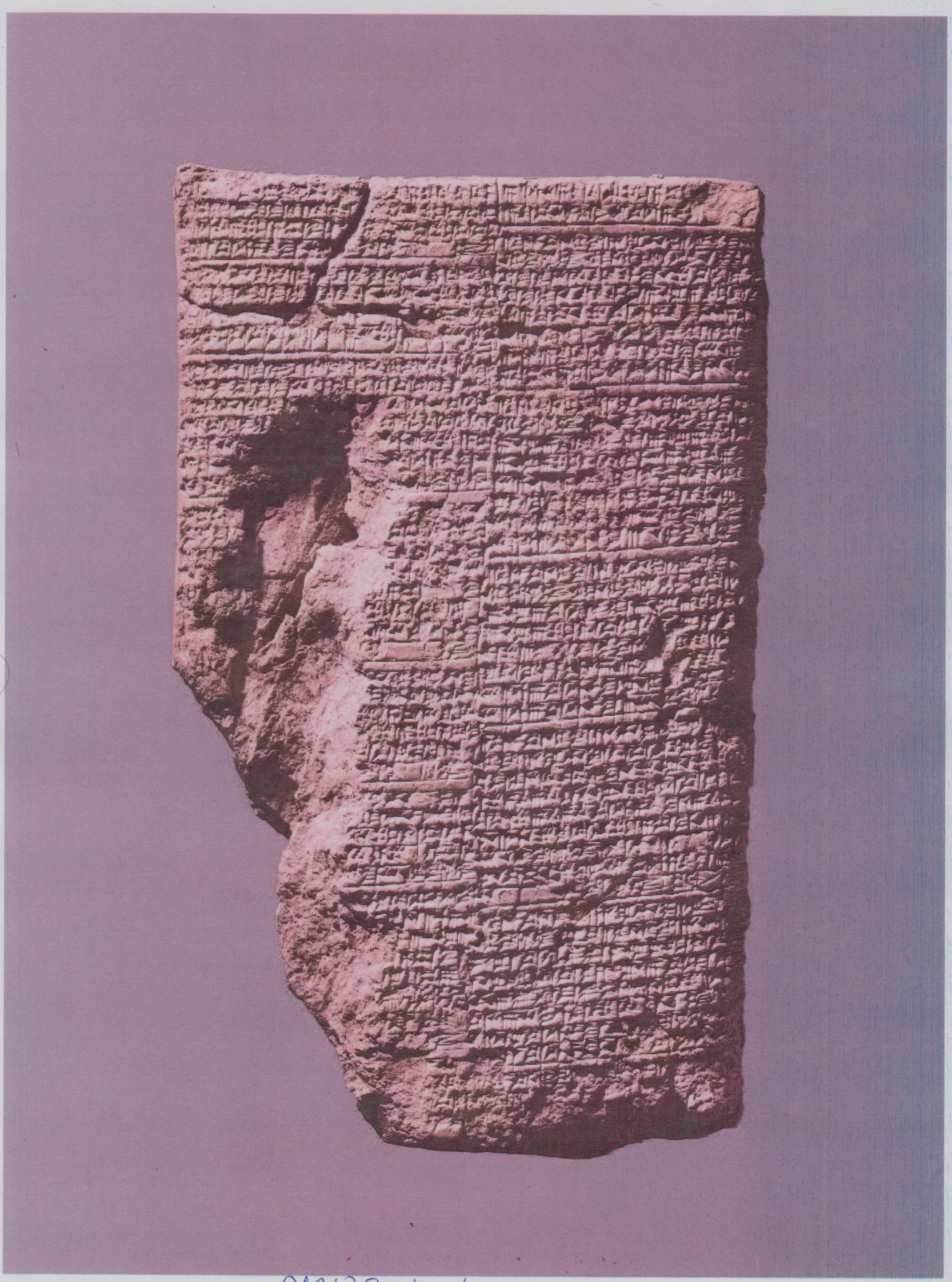
and along the way correctly summed consecutive square, basically had the equivalent of $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

- they didn't really in terms of angles, but in terms of slopes

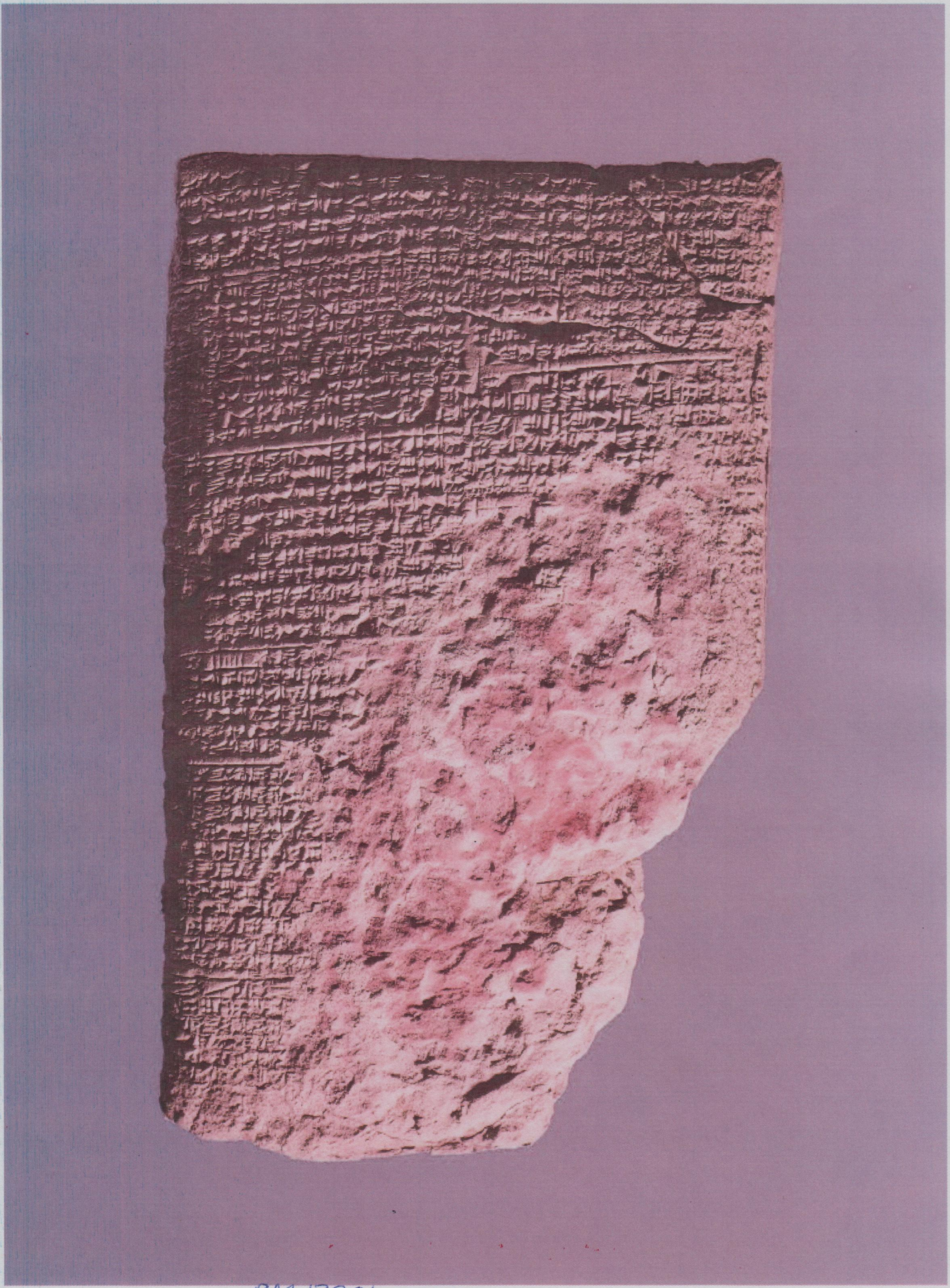


They'd work with $\frac{a}{b}$ ($= \tan(\theta)$) instead of θ.

We'll take a look in some detail at an algebra problem (from a tablet numbered BM13901) that has some geometric interpretation.



BM13901 obverse



BM 13901 reverse

The sixth problem with solution from the (somewhat damaged) cuneiform tablet BM13901, which probably dates to about 1800 B.C. somewhere in southern Mesopotamia.

I have added the area and two thirds of the side of my square and it is 0;35. You take 1, the "coefficient". Two thirds of 1, the coefficient, is 0;40.* Half of this, 0;20, you multiply by 0;20 [and the result] 0;6,40 you add to 0;35, and [the result] 0;41,40 has 0;50 as its square root. 0;20, which you multiplied by itself, you subtract from 0;50 and 0;30 is the [side of] the square.

* That two thirds of one is 40, tells us what scale we are using here as $\frac{2}{3} = \frac{40}{60}$, so ~~t~~ we know where to put the sexagesimal points.

So what is going on here?

Here is the arithmetic being done, in modern notation:

$\underline{1^\circ}$ $1 \cdot \frac{2}{3} = \frac{40}{60}$ $\underline{2^\circ}$ $\frac{1}{2} \cdot \frac{40}{60} = \frac{20}{60}$ $\underline{3^\circ}$ $\left(\frac{20}{60}\right)^2 = \frac{400}{3600} = \frac{6}{60} + \frac{40}{60^2}$

$\underline{4^\circ}$ $\left(\frac{6}{60} + \frac{40}{60^2}\right) + \frac{35}{60} = \frac{41}{60} + \frac{40}{60^2} = \frac{2500}{3600}$

$\underline{5^\circ}$ $\sqrt{\frac{41}{60} + \frac{40}{60^2}} = \sqrt{\frac{2500}{3600}} = \frac{50}{60}$ $\underline{6^\circ}$ $\frac{50}{60} - \frac{20}{60} = \frac{30}{60}$

In algebraic terms, we are trying to solve the equation $x^2 + \frac{2}{3}x = \frac{35}{60}$, where x is the side of a square, for x. We would likely do this using the general quadratic formula: $ax^2+bx+c=0 \underset{(\text{if } a \neq 0)}{\Longrightarrow} x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Let's see what is being done in this solution in terms of $ax^2+bx+c=0$, where $a=1$, $b=\frac{2}{3}$, and $c=-\frac{35}{60}$.

1° Multiply a times b to get ab.

2° Multiply ab by $\frac{1}{2}$ to get $\frac{ab}{2}$.

3° Square $\frac{ab}{2}$ to get $\left(\frac{ab}{2}\right)^2 = \frac{a^2b^2}{4}$.

4° Add $-c$ to $\frac{a^2b^2}{4}$ to get $\frac{a^2b^2}{4} - c$.

5° Take the square root of $\frac{a^2b^2}{4} - c$ to get $\sqrt{\frac{a^2b^2}{4} - c}$.

6° Subtract $\frac{ab}{2}$ from $\sqrt{\frac{a^2b^2}{2} - c}$ to get $x = \sqrt{\frac{a^2b^2}{4} - c} - \frac{ab}{2}$.

Does this work? If $a = 1$, $b = \frac{2}{3}$, & $c = \frac{-35}{60} = \frac{-7}{12}$,

then $x = \sqrt{\frac{1^2 \cdot \left(\frac{2}{3}\right)^2}{4} - \left(-\frac{35}{60}\right)} - \frac{1 \cdot 2/3}{4} = \sqrt{\frac{4/9}{4} + \frac{7}{12}} - \frac{2/3}{2}$

$= \sqrt{\frac{1}{9} + \frac{7}{12}} - \frac{1}{3} = \sqrt{\frac{4}{36} + \frac{28}{36}} - \frac{1}{3} = \sqrt{\frac{25}{36}} - \frac{1}{3}$

$= \frac{5}{6} - \frac{1}{3} = \frac{5}{6} - \frac{2}{6} = \frac{3}{6} = \frac{1}{2} = 0;30 = \frac{30}{60}$,

so the procedure works in <u>this</u> case. [as a=1..] However, in general, the correct solution would be $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} + \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$

$= \frac{-b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$. $\neq$ to

It turns out that one of the damaged bits of the tablet is where the first step "Multiply a times b to get ab" is given. If the reconstructors got it wrong and it should have read something like "Divide b by a to get $\frac{b}{a}$" [Well, really, "Divide $\frac{2}{3}$ by 1 to get $\frac{2}{3}$".], then the general procedure would be more like:

1° Divide b by a to get $\frac{b}{a}$.

2° Multiply $\frac{b}{a}$ by $\frac{1}{2}$ to get $\frac{b}{2a}$

3° Square $\frac{b}{2a}$ to get $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

4° Add $-c$ to $\frac{b^2}{4a^2}$ to get $\frac{b^2}{4a^2} - c$.

5° Take the square root of $\frac{b^2}{4a^2} - c$ to get $\sqrt{\frac{b^2}{4a^2} - c}$

6° Subtract $\frac{b}{2a}$ from $\sqrt{\frac{b^2}{4a^2} - c}$ to get

$$x = \sqrt{\frac{b^2}{4a^2} - c} - \frac{b}{2a} = \frac{1}{2a}\sqrt{b^2 - 4a^2c} - \frac{b}{2a} = \frac{-b + \sqrt{b^2 - 4a^2c}}{2a}$$

... which is almost correct; you need "$-4ac$" in the square root instead of "$-4a^2c$". This could be fixed by adding $\frac{-c}{a}$ in step 4 instead.

Moral: It's very hard to figure out what the author of BM13901 was capable of, from this problem alone, anyway. Two small changes to procedure given in the reconstructed solution would give an algorithm capable of solving arbitrary quadratic equations of the form $ax^2+bx=c$ for $a,b,c>0$. As it is, all we can more or less safely conclude is that the author could get the right answers for equations of the form $x^2+bx=c$ for $b,c>0$, ie when $a=1$. Either way, we can conclude the author could handle *some* quadratic equations.

Next time: start of Greek math!