Mesopotamian arithmetic and algebra

Notation: Instead of using cuneioform signs for sexagesimal numbers we'll use modern numbers in a similar way

𒌋𒌋𒐊 𒐈 $= 20 + 5 + \frac{3}{60} = 25\frac{1}{20}$

in modern sexagesimal : $25;3$ is $25 + \frac{3}{60}$

↑ sexagesimal point

Separate sexagesimal places by commas:

eg $85\frac{1}{20}$

𒁹𒌋𒌋𒐊 𒐈 $1,25;3$

Addition was done as we would carrying 𒁹's (1s) into 𒌋's (10s) as necessary and carrying 60's over to the next sexagesimal place. No symbol for plus - said it in words.

(Subtraction was done like in Egypt $a - b = c$ would be thought of as add to b until you get a)

Multiplication was also done as we'd do it, (before calculators!) with the help of multiplication tables

eg we have surviving tables of multiplying by 2 & other similar ones

1×2
2×2
3×2
$\vdots$
10×2

20×2
30×2
$\vdots$

& eg $25 \times 2 = 20 \times 2 + 5 \times 2$

Division was done by multiplying by reciprocals.

eg $54/4 = 54 \times \frac{1}{4}$ ...

This also had tables to speed things up: tables of reciprocals & tables of multiplications by reciprocals.

They did try to avoid using reciprocals such as $\frac{1}{7}$ that do not get finite representations in a sexagesimal system.

When they really had to, the usually used sexagesmal approximations to these reciprocals.

Similarly they used approximations for irrational numbers

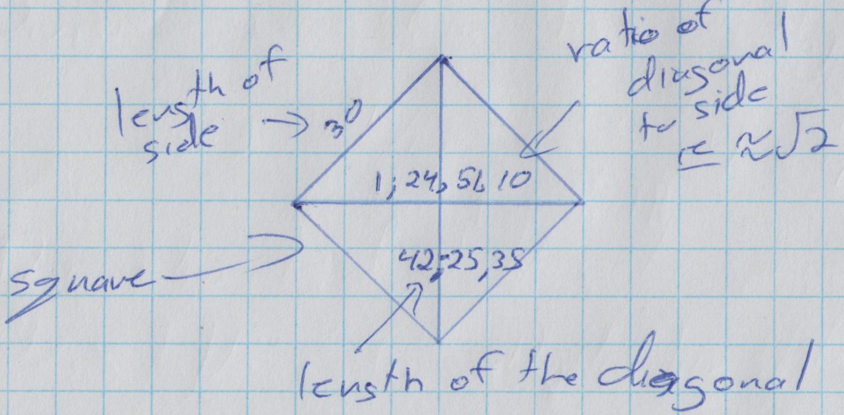
es a computation of a circular area that amounts to having $\pi \approx 3\frac{1}{8} = \frac{25}{8}$

Also $\sqrt{2}$ turns up in a couple of approximations

1) Quick and dirty: $\sqrt{2} \approx 1;25 = 1 + \frac{25}{60} = \frac{85}{60} = \frac{17}{12} \approx 1.416$

$\sqrt{2} = 1.414213\ldots$

2) On the tablet YBC 7289



$1;24,51,10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$

≈ 1.414214

(very close to the actual value of $\sqrt{2}$)

How did they get these approximations?
Trial and error? They might have had a method similar to Heron's method.

Heron's method for computing $\sqrt{a}$ for a non-square positive a.

Find an a_0 which is rational s.t. $\sqrt{a} < a_0 < \sqrt{a+1}$.

Then let, given a_n, $a_{n+1} = (a_n + a/a_n)/2$.

When you take the limit: $\lim_{n\to\infty} a_{n+1} = \lim_{a_n\to\infty} (a_n + a/a_n)/2 = A$.

$$= (A + \tfrac{a}{A})/2$$

$$\Rightarrow A = (A + \frac{a}{A})/2$$

$$\Rightarrow 2A = A + \frac{a}{A} \Rightarrow A = \frac{a}{A} \Rightarrow A^2 = a$$

ie $A = \sqrt{a}$.

Eg for computing $\sqrt{2}$, if you start with $a_0 = \frac{3}{2}$

then $\underset{1.4}{\sqrt{2}} < \underset{1.5}{\frac{3}{2}} < \underset{1.7}{\sqrt{3}}$

$$a_1 = \left(\frac{3}{2} + \frac{2}{3/2}\right)/2 = \left(\frac{3}{2} + \frac{4}{3}\right)/2 = \frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$

$$a_2 = \left(\frac{17}{12} + \frac{2}{17/12}\right)/2 = \frac{17}{24} + \frac{12}{17} = \frac{289}{408} + \frac{288}{408} = \frac{577}{408} \approx 1.414215\ldots$$

From AO8862, a tablet from c.1750 B.C.

Length, width. I have multiplied the length and the width, obtaining the area. Then I added to the area, the excess of length over the width: 183 was the result. I have added the length and the width: 27 was the result. Required length, width, and area.

~~This is the method: [using modern numbers]~~

27 and 183, the sums; 15 the length; 180 the area; 12 the width.

This is the method: [using modern notation]

$27 + 183 = 210$, $2 + 27 = 29$

Take $\frac{1}{2}$ of 29, this gives $14\frac{1}{2}$.

$14\frac{1}{2} \times 14\frac{1}{2} = 210\frac{1}{4}$,

$210\frac{1}{4} - 210 = \frac{1}{4}$.

In our terms:

Let x = length & y = width.

$$xy + (x-y) = 183$$

$$x + y = 27$$

Solve for x, y, & Area $= xy$.

Solution:

$x = 15$, $y = 12$, $A = xy = 180$

How we get the solution:

$(x+y) + (xy + x - y) = 27 + 183 = 210$

$= xy + 2x = \cancel{(x+2)}\ \cancel{x(x+2)} = x(y+2)$

Let $v = y+2$, so $xv = 210$

and $x + v = x + y + 2 = 27 + 2 = 29$.

The square root of $\frac{1}{4}$ is $\frac{1}{2}$.

$14\frac{1}{2}+\frac{1}{2} = 15$, the length;

$14\frac{1}{2}-\frac{1}{2} = 14$, the width.

Subtract 2, which has been added to 27, from 14, the width. 12 is the actual width. I have multiplied the length 15 by the width 12.

$15 \times 12 = 180$, the area.

$15-12=3$; $180+3=183$.

Thus $\frac{x+v}{2} = \frac{29}{2} = 14\frac{1}{2}$ (6)

and $\left(\frac{x+v}{2}\right)^2 = \frac{x^2+2xv+v^2}{4} = (14\frac{1}{2})^2 = 210\frac{1}{4}$.

Thus $\frac{x^2-2xv+v^2}{4} = \left(\frac{x+v}{2}\right)^2 - xv = \left(\frac{x-v}{2}\right)^2$

[the discriminant] $= 210\frac{1}{4} - 210 = \frac{1}{4}$

Thus $\frac{x-v}{2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, so

$x = \frac{x-v}{2} + \frac{x+v}{2} = \frac{1}{2} + 14\frac{1}{2} = 15$

and $v = \frac{x+v}{2} - \frac{x-v}{2} = 14\frac{1}{2} - \frac{1}{2} = 14$

so $y+2 = v = 14$, and so $y = 12$.

$A = xy = 15 \times 12 = 180$.

They could solve some fairly challenging equations by methods we can follow.