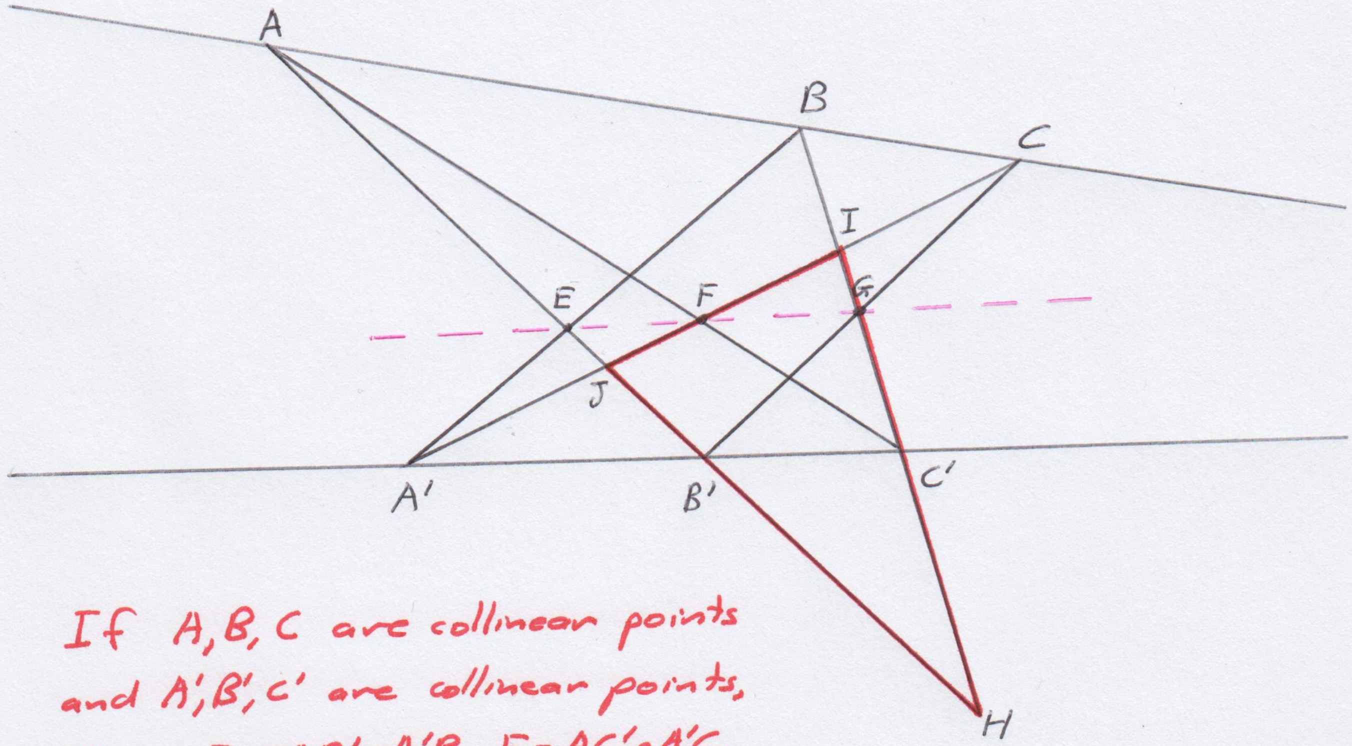
Pappus' Theorem

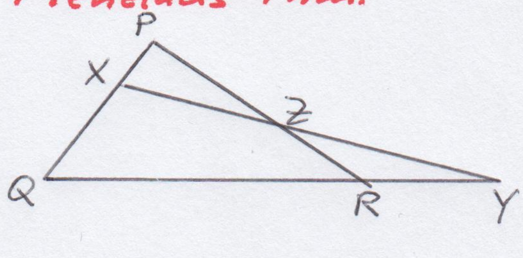
(a.k.a. Pappus' Hexagon Thm.)
[from Book VII of the *Collection*, spread out over Propositions 138–143.]



If A, B, C are collinear points and A', B', C' are collinear points, then $E = AB' \cap A'B$, $F = AC' \cap A'C$, and $G = BC' \cap B'C$ are also collinear.

proof: We will use the modern version of Menelaus' Thm. ...

Recall: Menelaus' Thm.



Point X, Y, Z on (extensions of) the sides PQ, QR, RP of ΔPQR are collinear iff

$$\frac{PX}{XQ}\cdot\frac{QY}{YR}\cdot\frac{RZ}{ZP} = -1.$$

(Using the convention that $AB = -BA$.)

... six times over!

Extend AB' and BC' until they meet at H and consider ΔHIJ.

By Menelaus' Thm. applied to ΔHIJ:

1) A', E, B are collinear, so $\frac{HB}{BI} \cdot \frac{IA'}{A'J} \cdot \frac{JE}{EH} = -1$.

2) A, F, C' —— " —— $\frac{HC'}{C'I} \cdot \frac{IF}{FJ} \cdot \frac{JA}{AH} = -1$.

3) C, G, B' —— " —— $\frac{HG}{GI} \cdot \frac{IC}{CJ} \cdot \frac{JB'}{B'H} = -1$.

4) A, B, C —— " —— $\frac{HB}{BI} \cdot \frac{IC}{CJ} \cdot \frac{JA}{AH} = -1$.

5) A', B', C' —— " —— $\frac{HC'}{C'I} \cdot \frac{IA'}{A'J} \cdot \frac{JB'}{B'H} = -1$

} We will be using these two in inverted form.

It follows that $-1 = (-1)^5$

$$= \left(\cancel{\frac{HB}{BI}} \cdot \cancel{\frac{IA'}{A'J}} \cdot \frac{JE}{EH}\right)\left(\cancel{\frac{HC'}{C'I}} \cdot \frac{IF}{FJ} \cdot \cancel{\frac{JA}{AH}}\right)\left(\frac{HG}{GI} \cdot \cancel{\frac{IC}{CJ}} \cdot \frac{JB'}{B'H}\right)\left(\cancel{\frac{BI}{HB}} \cdot \cancel{\frac{CJ}{IC}} \cdot \cancel{\frac{AH}{JA}}\right)\left(\cancel{\frac{C'I}{HC'}} \cdot \cancel{\frac{A'J}{IA'}} \cdot \frac{B'H}{JB'}\right)$$

$$= \frac{JE}{EH} \cdot \frac{IF}{FJ} \cdot \frac{HG}{GE},$$ so, by the other direction of Menelaus' Theorem, E, F, G are collinear. //

Pappus is also known for his Centroid Theorems:

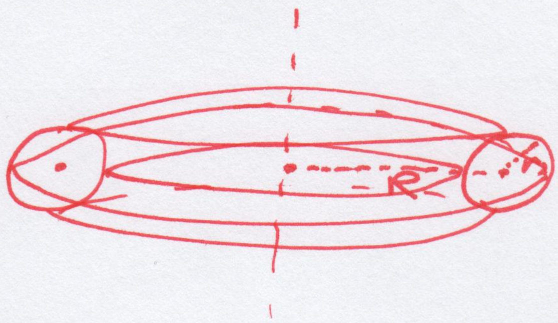
<u>Pappus' First Centroid Theorem</u>

The surface area of the surface of revolution generated by revolving a curve C about an axis external to C (but in the same plane as C) is $SA = sd$, where s is the arc-length of C and $d = 2\pi r$ is the distance traveled by the geometric centroid of C.

Pappus' Second Centroid Theorem

The volume of the solid of revolution generated by revolving a plane figure F about an axis external to F (but in the same plane as F) is $V = Ad$, where A is the area of F and $d = 2\pi r$ is the distance traveled by the geometric centroid of F.

eg.



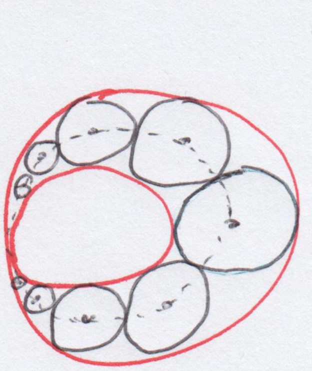
Suppose $r < R$ and we rotate a circle of radius r about an axis that R away from the centre of the circle to make a torus.

Then $\quad SA = (2\pi R)(2\pi r) = 4\pi^2 Rr$

and $\quad V = (2\pi R)(\pi r^2) = 2\pi^2 Rr^2$

for this torus.

A rather more obscure result is Pappus' Chain:



If you start with two circles tangent at one point, and one inside the other, and then draw circles tangent to both and to each other, the centres of the new circles are on a common ellipse.