Heron of Alexandria (c. 10 – 70 A.D.)

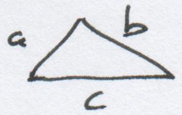
(often called Hero in translation)

- apparently a native of Alexandria, likely taught at the Museum
- primarily an engineer and mathematician
- as an engineer he is especially interesting for designing machines, including:
  - a primitive steam engine consisting of a vessel of boiling water mounted on an axle, with nozzles expelling the steam causing the vessel to spin
  - a windmill powering an organ
  - various mechanisms for creating sound and other effects in a theatre
  - a vending machine (!)
  - a piston pump
- as a mathematician he is mainly remembered for two results:
  - Heron's method (also known as the Babylonian method) for computing the square root of a number a:

    1) Guess an approximation a_0 to $\sqrt{a}$.
    2) Given a_n, let $a_{n+1} = \dfrac{a_n + \frac{a}{a_n}}{2} = \dfrac{a_n^2 + a}{2a_n}$.
    3) Repeat step 2 until one has the precision one desires

    This is believed to be the method used by the Mesopotamian civilizations to compute square roots, but Heron is the earliest to describe the method, and have his description survive.

  - Heron's formula for computing the area of a triangle:

    Given a triangle with sides of length a, b, & c, respectively, let the semi-perimeter of the triangle be

    $$s = \frac{a+b+c}{2}.$$

    Then the area of the triangle is

    $$\sqrt{s(s-a)(s-b)(s-c)}.$$

    (This formula was later extended to cyclic quadrilaterals by the Indian mathematician Brahmagupta.)

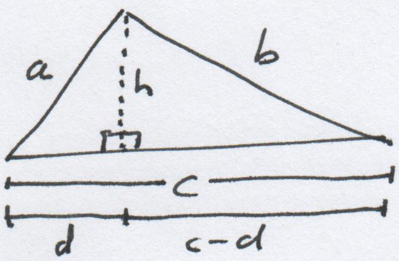
- Why does Heron's method work?

Well, if $a_{n+1} = \dfrac{a_n^2 + a}{2a_n}$ for all $n \geq 0$,

then $\alpha = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n^2+a}{2a_n} = \dfrac{\alpha^2+a}{2\alpha}$,

so $2\alpha^2 = \alpha^2 + a$, ie $\alpha^2 = a$, so $\alpha = \sqrt{a}$. //

- Why does Heron's formula work?



Let h be the altitude of the triangle from the vertex opposite side c. Then the area of the triangle is $\frac{1}{2}ch$.

The foot of the altitude divides side c into pieces of length d & (below side a) and length $c-d$ (below side b).

By the Pythagorean Theorem we have $h^2+d^2=a^2$ and $h^2+(c-d)^2=b^2$. It follows that
$$h^2 = a^2-d^2 = b^2-(c-d)^2,$$
so
$$a^2-d^2 = b^2-(c-d)^2 = b^2-(c^2-2cd+d^2) = b^2-c^2+2cd-d^2.$$
It follows in turn that $a^2 = b^2-c^2+2cd$, so $d = \dfrac{a^2-b^2+c^2}{2c}$.

Since $h^2 = a^2-d^2$, we have
$$\begin{aligned}
h^2 = a^2-d^2 &= a^2 - \left(\frac{a^2-b^2+c^2}{2c}\right)^2 \\
&= \left(\frac{2ac}{2c}\right)^2 - \left(\frac{a^2-b^2+c^2}{2c}\right)^2 \\
&= \frac{(2ac+a^2-b^2+c^2)(2ac-a^2+b^2-c^2)}{4c^2} \\
&= \frac{[(a+c)^2-b^2][b^2-(a-c)^2]}{4c^2} \\
&= \frac{(a+c+b)(a+c-b)(b-a+c)(b+a-c)}{4c^2} \\
&= \frac{2s\cdot 2(s-b)\cdot 2(s-a)\cdot 2(s-c)}{4c^2} = \frac{4s(s-a)(s-b)(s-c)}{c^2}
\end{aligned}$$

Thus the area of the triangle is

$$\frac{1}{2}ch = \frac{1}{2}c\sqrt{h^2} = \frac{1}{2}c\sqrt{\frac{4s(s-a)(s-b)(s-c)}{c^2}}$$

$$= \frac{\cancel{c}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{c}}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \quad . \qquad //$$

Both Heron's method and Heron's formula occur in his work Metrica, which is concerned with computing the area and volumes of various shapes. Two other surviving books of his have substantial mathematical content: On the Dioptra, which is concerned with methods of measuring lengths and distances and includes descriptions of an odometer and the eponymous dioptra, a surveying instrument that is a distant ancestor of the theodolite used today. The other is the Catoptrica, a work on light and optics, particularly mirrors. It includes a statement of the "least path principle", that if a ray of light propagates between two points in the same medium, then it follows the shortest path between the two points.

(Most of Heron's surviving works did so at least in part in Arabic translation.)