Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2015

Take-home Final Exam
Due on Wednesday, 22 April, 2014.

Instructions: Do all three of parts $\nearrow$, $\searrow$, and $\swarrow$, and, if you wish, part $\nwarrow$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part $\nearrow$. Do *all* four (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Define a sequence $\{c_n\}$ by setting $c_0 = 1$ and $c_{n+1} = \dfrac{1}{1+c_n}$ for $n \geq 0$. Show that $\displaystyle\lim_{n\to\infty} c_n$ exists and find its value.

2. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n!e^n}{(2n)!}x^n$.

3. Is $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ uniformly continuous on $(-1, 1)$ or not?

4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ converges uniformly on $(-\infty, \infty)$ or not.

Part $\searrow$. Do any *two* (2) of problems **5** – **8**. *[20 = 2 × 10 each]*

5. Determine the radius of convergence of the power series

$$1 + \frac{1}{2}x + \frac{1}{2}x^3 + x^4 + \frac{1}{2}x^5 + \frac{1}{2}x^7 + x^8 + \frac{1}{2}x^9 + \frac{1}{2}x^{11} + x^{12} + \frac{1}{2}x^{13} + \frac{1}{2}x^{15} + \cdots$$

and find the function of which it is the Taylor series.

6. Verify that $f(x) = \arctan(x)$ is uniformly continuous on $(-\infty, \infty)$.

7. Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely. Show that $\displaystyle\sum_{n=0}^{\infty} a_n^2$ converges.

8. Suppose $\{a_k\}$ is a sequence which has no convergent subsequences. What can you deduce about $\displaystyle\liminf_{k\to\infty} a_k$ and $\displaystyle\limsup_{k\to\infty} a_k$?

Part ↙. Do any *two* (2) of problems **9** – **12**. *[20 = 2 × 10 each]*

9. Show that $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at $x = 0$, but is not continuous at any other point.

10. Compute $\lim_{n \to \infty} (n!)^{1/n}$.

11. Let $E = \{\, r \in (0,1) \mid \textit{the decimal expansion of } r \textit{ uses only the digits } 0,\ 2,\ 4,\ 6,\ \&\ 8 \,\}$. Find $\sup(E)$ and $\inf(E)$ and show that there are r and s in E with $r < s$ for which there is no $t \in E$ with $r < t < s$.

12. Suppose α is a real number not in $\mathbb{Z}^{\leq 0} = \{\, 0, -1, -2, \dots \}$, and consider the following power series:

$$\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!} x^n = \frac{\alpha}{1} x + \frac{\alpha(\alpha+1)}{1 \cdot 2} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

Determine for which values of x this series diverges, converges conditionally, and converges absolutely, respectively.

Part ↖.

↻. Write a poem about real analysis or mathematics in general. *[2]*

[Total = 80]

I HOPE THAT YOU ENJOYED THIS COURSE.
ENJOY THE SUMMER EVEN MORE!