Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2014

Solutions to Assignment #1
Basic epsilonics

This assignment is a warm-up using something that you should have seen some version of in first-year caculus, the ε–δ definition of limits. Please look it up in our present text or in your old calculus textbook!

1. Use the ε–δ definition of limits to verify that $\lim_{x \to 13} (2x - 3) = 23$. *[2]*

SOLUTION. Suppose $\varepsilon > 0$ is given. We try to reverse-engineer the necssary $\delta > 0$:

$$|(2x-3)-23| < \varepsilon \;\Leftrightarrow\; |2x-26| < \varepsilon \;\Leftrightarrow\; 2|x-13| < \varepsilon \;\Leftrightarrow\; |x-13| < \frac{\varepsilon}{2}$$

Setting $\delta = \frac{\varepsilon}{2}$, we get that if $|x-13| < \delta = \frac{\varepsilon}{2}$, then $|(2x-3)-23| < \varepsilon$. [To see this, note that the implications in our reverse-engineering of δ run both ways.] Thus, by the ε–δ definition of limits, $\lim_{x \to 13} (2x-3) = 23$. ■

2. Use the ε–δ definition of limits to verify that $\lim_{x \to 2} x^2 = 4$. *[3]*

SOLUTION. Suppose $\varepsilon > 0$ is given. We try to reverse-engineer the necessary $\delta > 0$:

$$\left|x^2 - 4\right| < \varepsilon \;\Leftrightarrow\; |x-2| \cdot |x+2| < \varepsilon \;\Leftrightarrow\; |x-2| < \frac{\varepsilon}{|x+2|}\,,$$

just so long as $x + 2 \neq 0$, *i.e.* as long as $x \neq -2$. Besides the problem that we must ensure that $x \neq -2$, we also cannot have δ depend on x. We can solve both of these problems by only accepting δs small enough to ensure that if $|x-2| < \delta$, then $x \neq -2$ with some margin to spare. For example, suppose we accept only $0 < \delta \leq 1$. If $|x-2| < \delta \leq 1$, then

$$\begin{aligned} |x-2| < 1 \;&\Rightarrow\; -1 < x-2 < 1 \;\Rightarrow\; 1 < x < 3 \\ &\Rightarrow\; 3 < x+2 < 5 \;\Rightarrow\; \frac{1}{3} > \frac{1}{x+2} = \frac{1}{|x+2|} > \frac{1}{5}\,. \end{aligned}$$

It follows that if $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$, then, whenever $|x-2| < \delta$, we have $|x-2| < 1$, and so $|x-2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|}$, from which it follows that $\left|x^2-4\right| < \varepsilon$, by our initial attempt at reverse-engineering δ. Thus, by the ε–δ definition of limits, $\lim_{x \to 2} x^2 = 4$. ■

3. Use the ε–δ definition of limits to verify that $\lim_{x \to c} x^2 = c^2$ for every real number c. *[4]*

Hint: You may find it useful to consider the cases $c = 0$ and $c \neq 0$ separately in doing **3**.

SOLUTION. The method used in the solution to question 2 above will work here, though a little care must be taken to avoid accidentally dividing by 0 and the like.

Suppose $\varepsilon > 0$ is given. We try to reverse-engineer the necessary $\delta > 0$:

$$\left|x^2 - c^2\right| < \varepsilon \iff |x-c| \cdot |x+c| < \varepsilon \iff |x-c| < \frac{\varepsilon}{|x+c|} \,,$$

just so long as $x + c \neq 0$, *i.e.* as long as $x \neq -c$. Again, we also have the problem that δ cannot depend on x. Note that we cannot proceed naively as in the solution to 2 by only accepting δs small enough to ensure that if $|x-c| < \delta$, then $x \neq -c$ with some margin to spare if $c = 0 = -0$. We will therefore assume that $c \neq 0$ and then take care of the case that $c = 0$ separately.

i. In the case that $c \neq 0$, we will accept only $0 < \delta \leq |c|$. If $|x-c| < \delta \leq |c|$, then

$$\begin{aligned} |x-c| < |c| &\Rightarrow -|c| < x - c < |c| \Rightarrow c - |c| < x < c + |c| \\ &\Rightarrow 2c - |c| < x + c < 2c + |c| \,. \end{aligned}$$

Note that it is possible here that $x + c < 0$, and, indeed, that $2c - |c| < 0$, so we don't have quite as easy a time as in the solution to 2 even in this case. However, whether the items in the last inequality of our reverse-engineering attempt above are negative or positive, it is still true (assuming $|x-c| < |c|$) that it follows that $|c| < |x+c| < 3|c|$, and hence that $\frac{1}{|c|} > \frac{1}{|x+c|} > \frac{1}{3|c|}$. (Recall that $c \neq 0$ in this case ...)

It follows that if $\delta = \min\left(|c|, \frac{\varepsilon}{3|c|}\right)$, then, whenever $|x-c| < \delta$, we have $|x-c| < |c|$, and so $|x-c| < \frac{\varepsilon}{3|c|} < \frac{\varepsilon}{|x+c|}$, from which it follows that $\left|x^2 - c^2\right| < \varepsilon$, by our initial attempt at reverse-engineering δ. $\square$

ii. In the case that $c = 0$, we have

$$\left|x^2 - 0^2\right| < \varepsilon \iff \left|x^2\right| < \varepsilon \iff |x| < \sqrt{\varepsilon} \iff |x - 0| < \sqrt{\varepsilon} \,.$$

Since every implication above is reversible, it follows that of we take $\delta = \sqrt{\varepsilon}$, then whenever $|x-0| < \sqrt{\varepsilon}$, we get $\left|x^2 - 0^2\right| < \varepsilon$, as required. $\square$

Thus, no matter what $c \in \mathbb{R}$ we may have, for very $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x-c| < \delta$, then $\left|x^2 - c^2\right| < \varepsilon$. Hence, by the ε–δ definition of limits, $\lim_{x \to c} x^2 = c^2$ for every $c \in \mathbb{R}$. $\blacksquare$