Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2014
Solutions to the Quizzes

Quiz #1. Tuesday, 14 January, 2014. *[10 minutes]*

1. Suppose you are given that $\inf\left\{\frac{1}{n} \mid n \in \mathbb{N} \text{ and } n > 0\right\} = 0$. Use this fact to help prove the Archimedean Property of $\mathbb{R}$, *i.e.* that $\mathbb{N}$ has no upper bound in $\mathbb{R}$. *[5]*

SOLUTION. Suppose, by way of contradiction, that $u \in \mathbb{R}$ was an upper bound for $\mathbb{N}$, *i.e.* $n < u$ for all $n \in \mathbb{N}$.

Since $0 < u$ (remember that $0 \in \mathbb{N}$), we have that $0 < \frac{1}{u}$. [The reciprocal of a positive number is also positive.]

On the other hand, since $n < u$ for all $n \in \mathbb{N}$ with $n > 0$, we have $\frac{1}{n} > \frac{1}{u}$ for all such n. [Taking reciprocals reverses inequalities.] It follows that $\frac{1}{u}$ is a lower bound for $\left\{\frac{1}{n} \mid n \in \mathbb{N} \text{ and } n > 0\right\}$. Since we are given that $\inf\left\{\frac{1}{n} \mid n \in \mathbb{N} \text{ and } n > 0\right\} = 0$, it follows that $\frac{1}{u} \leq 0$.

Thus $0 < \frac{1}{u} \leq 0$, so $0 < 0$, which is impossible. Since assuming otherwise led to a contradiction, $\mathbb{N}$ has no upper bound in $\mathbb{R}$. ■

Quiz #2. Tuesday, 21 January, 2014. *[10 minutes]*

1. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences with $\lim\limits_{n\to\infty} s_n = 3$ and $\lim\limits_{n\to\infty} (t_n - s_n) = 0$. Use the ε–N definition of the limit of a sequence to show that $\lim\limits_{n\to\infty} t_n = 3$, too. *[5]*

SOLUTION. $\lim\limits_{n\to\infty} s_n = 3$ means that $\forall \alpha > 0\ \exists M \in \mathbb{N}\ \forall m \geq M : |s_m - 3| < \alpha$. Similarly, $\lim\limits_{n\to\infty} (t_n - s_n) = 0$ means that $\forall \beta > 0\ \exists K \in \mathbb{N}\ \forall k \geq K : |(t_k - s_k) - 0| < \beta$.

We need to show that $\lim\limits_{n\to\infty} t_n = 3$, *i.e.* that $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : |t_n - 3| < \varepsilon$.

Suppose $\varepsilon > 0$. Observe that for all n,

$$|t_n - 3| = |t_n - s_n + s_n - 3| \leq |t_n - s_n| + |s_n - 3|\ ,$$

by the triangle inequality. Let $\alpha = \beta = \frac{\varepsilon}{2}$; then $\alpha = \beta > 0$. It follows that there is an $M \in \mathbb{N}$ such that for all $m \geq M$, $|s_m - 3| < \alpha$, as well as a $K \in \mathbb{N}$ such that for all $k \geq K$, $|(t_k - s_k) - 0| < \beta$. Let $N = \max(M, K)$. Then for all $n \geq N$ we have that $n \geq M$ and $n \geq K$, so

$$|t_n - 3| \leq |t_n - s_n| + |s_n - 3| < \beta + \alpha = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

as required.

Thus $\lim\limits_{n\to\infty} t_n = 3$. ■

Quiz #3. Tuesday, 28 January, 2014. *[10 minutes]*

1. Show that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$. [With or without epsilonics – your choice!] *[5]*

SOLUTION. [Without epsilonics, of course!] Recall that $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$; it follows that $-1 \leq \cos(n) \leq 1$ for all $n \in \mathbb{N} \subsetneq \mathbb{R}$, and hence that $-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. (Well, other than $n = 0$, which doesn't matter because ... :-) Since $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$, it follows by the Squeeze Theorem that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$. ■

Quiz #4. Tuesday, 4 February, 2014. *[10 minutes]*

1. Suppose that $\{s_{n_k}\}$ is a subsequence of the sequence $\{s_n\}$. Which of

$$\limsup_{n\to\infty} s_n \geq \limsup_{k\to\infty} s_{n_k} \quad \text{or} \quad \limsup_{n\to\infty} s_n \leq \limsup_{k\to\infty} s_{n_k}$$

must be true? Explain why! *[5]*

SOLUTION. Recall, from class or the textbook, that

$$\limsup_{n\to\infty} s_n = \inf \{ \beta \mid \exists N \forall n \geq N : s_n < \beta \}$$

and

$$\limsup_{k\to\infty} s_{n_k} = \inf \{ \beta \mid \exists K \forall k \geq K : s_{n_k} < \beta \} .$$

Suppose $\beta \in \mathbb{R}$ is such that for some N we have that for all $n \geq N$, $s_n < \beta$. Then there is a K such that for all $k \geq K$, $n_k \geq N$, and so $s_{n_k} < \beta$, too. Thus

$$\{ \beta \mid \exists N \forall n \geq N : s_n < \beta \} \subseteq \{ \beta \mid \exists K \forall k \geq K : s_{n_k} < \beta \} ,$$

and the greatest lower bound of a subset of a given set cannot be smaller than the greatest lower bound of the given set. (Think about it!) It follows that we must have

$$\begin{aligned} \limsup_{n\to\infty} s_n &= \inf \{ \beta \mid \exists N \forall n \geq N : s_n < \beta \} \\ &\geq \inf \{ \beta \mid \exists K \forall k \geq K : s_{n_k} < \beta \} = \limsup_{k\to\infty} s_{n_k} . \qquad \square \end{aligned}$$

Quiz #5. Tuesday, 11 February, 2014. *[10 minutes]*

1. Suppose $\sum_{n=1}^{\infty} |a_n|$ is a convergent series. Show that $\sum_{n=1}^{\infty} a_n$ converges as well. *[5]*

SOLUTION. We will use the Cauchy Convergence Criterion twice, once in each direction of the "if and only if" statement.

First, since $\sum_{n=1}^{\infty} |a_n|$ converges, it follows by the Cauchy Convergence Criterion that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $m \geq k \geq N$, then $\sum_{n=k+1}^{m} |a_n| < \varepsilon$.

Second, suppose an $\varepsilon > 0$ is given. If we choose N as above, then

$$\left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| < \varepsilon \,.$$

Thus the series $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy Convergence Criterion, and so converges. ■

Quiz #6. Tuesday, 25 February, 2014. *[10 minutes]*

1. Suppose $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. Does it follow that $\displaystyle\sum_{k=1}^{n} k a_k$ is bounded for all n? Prove it or give a counterexample. *[5]*

SOLUTION. Here is a counterexample: Let $a_k = \dfrac{1}{k^2}$ for $k \geq 1$. Then, on the one hand, $\displaystyle\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (*e.g.* by the p-Test, since $2 > 1$). On the other hand, $\displaystyle\sum_{k=1}^{n} k a_k = \sum_{k=1}^{n} \frac{k}{k^2} = \sum_{k=1}^{n} \frac{1}{k} \to \infty$ (and so is not bounded) as $n \to \infty$, since the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ is a series of positive terms that diverges. ■

Quiz #7. Tuesday, 4 March, 2014. *[10 minutes]*

1. It turns out that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln(2)$. If so, what is $\displaystyle\sum_{k=0}^{\infty} \left[\frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4}\right] = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$? *[5]*

Solution. Brute algebra and pattern recognition:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left[\frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4}\right] \\
&= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \ldots \\
&= \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right] = \frac{1}{2}\ln(2) \qquad \blacksquare
\end{aligned}$$

Quiz #8. Tuesday, 11 March, 2014. *[10 minutes]*

1. For each $n \geq 0$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \arctan(nx)$. Determine the function $f : \mathbb{R} \to \mathbb{R}$ that is the pointwise limit of the f_n (*i.e.* such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$), and whether it is continuous or not. *[5]*

Solution. There are three cases we need to consider, depending on whether $x < 0$, $x = 0$, or $x > 0$:

i. If $x < 0$, then $\displaystyle\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \arctan(nx) = \lim_{t\to-\infty} \arctan(t) = -\frac{\pi}{2}$.

ii. If $x = 0$, then $\displaystyle\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \arctan(n \cdot 0) = \lim_{n\to\infty} \arctan(0) = 0$.

iii. If $x > 0$, then $\displaystyle\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \arctan(nx) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$.

It follows that $f(x) = \begin{cases} -\frac{\pi}{2} & x < 0 \\ 0 & x = 0 \\ \frac{\pi}{2} & x > 0 \end{cases}$, which has a double jump discontinuity at 0, but is continuous everywhere else. $\blacksquare$

Quiz #9. Tuesday, 18 March, 2014. *[10 minutes]*

1. Suppose that for $n \geq 0$, $p_n(x) = a_n x^n$ for a sequence $\{a_n\}$ of positive real numbers such that $\lim_{n\to 0} a_n = 0$, and let $\zeta(x) = 0$ for all x. Show that $p_n \underset{\text{unif}}{\longrightarrow} \zeta$ on $[-1, 1]$. *[5]*

SOLUTION. We need to show that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in [-1, 1]$, $|p_n(x) - \zeta(x)| < \varepsilon$.

Suppose that an $\varepsilon > 0$ is given. Choose N such that if $n \geq N$, then $|a_n - 0| = a_n < \varepsilon$. Then, for all $n \geq N$ and all $x \in [-1, 1]$, we have

$$|p_n(x) - \zeta(x)| = |a_n x^n - 0| = a_n |x|^n \leq a_n 1^n = a_n < \varepsilon\,.$$

It follows by definition that $p_n \underset{\text{unif}}{\longrightarrow} \zeta$ on $[-1, 1]$. ■

Quiz #10. Tuesday, 18 March, 2014. *[15 minutes]*

1. Find a power series equal to $f(x) = \dfrac{1}{(1-x)^2}$ (when it converges) and determine its interval of convergence. *[5]*

   Hint: $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$. Mind you, there is at least one completely different way to get the series ...

SOLUTION. We'll use the hint and the facts that, by the formula for the sum of a geometric series, $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$, and that a power series may be differentiated term-by-term within its radius of convergence:

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots + x^n + \cdots\right)\\ &= \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{dx}x^2 + \frac{d}{dx}x^3 + \cdots + \frac{d}{dx}x^n + \cdots\\ &= 0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

It remains to determine the interval of convergence of this power series. First, note that the radius of convergence must be the same as that of the parent geometric series, namely $R = 1$. (This can also be obtained directly using the Ratio or the Root Test.) Second, at the endpoints $x = \pm 1$, we have

$$\lim_{n\to\infty} \left|n(\pm 1)^{n-1}\right| = \lim_{n\to\infty} n = \infty \neq 0\,,$$

so the series diverges at both endpoints by the Divergence Test. ■

Quiz #11. Wednesday, 2 April, 2014. *[20 minutes]*

1. Use Taylor's formula to find the Taylor series at 0 of $f(x) = e^{x/2}$. *[3]*
2. Show that the Taylor series at 0 of $f(x)$ converges (pointwise) to $f(x)$ for all x. *[2]*

   Hint: The Lagrange form of the nth remainder term for the Taylor series at 0 is $R_n(x) = \frac{f^{(n)}(t)}{n!}x^n$, where t is between 0 and x.

Solution to 1. We'll take some derivatives and evaluate them at 0. Note that $\frac{d}{dx}e^{x/2} = e^{x/2}\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}e^{x/2}$ and that $e^{0/2} = e^0 = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^{x/2}$	1
1	$\frac{1}{2}e^{x/2}$	$\frac{1}{2}$
2	$\frac{1}{4}e^{x/2}$	$\frac{1}{4}$
3	$\frac{1}{8}e^{x/2}$	$\frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2^n}e^{x/2}$	$\frac{1}{2^n}$
$\vdots$	$\vdots$	$\vdots$

Plugging this into Taylor's formula at 0 gives:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{\frac{1}{2^n}}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{2^n n!} \qquad \square$$

Solution to 2. By definition, $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \lim_{k\to\infty} T_n(x)$, where $\displaystyle T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$, and so the series converges to $f(x)$ for some x when $\displaystyle\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - t_n(x)] = 0$. Following the hint, we will use the Lagrange form of the remainder. Note that in this case $f^{(k)}(t) = \frac{1}{2^k}e^{t/2}$; since $e^{x/2}$ is a positive and increasing function, it follows that if t is between 0 and x, then $e^{t/2} \le e^{|x|/2}$ even if x is negative. This means that

$$0 \le |R_n(x)| = \left|\frac{f^{(n)}(t)}{n!}x^n\right| = \left|\frac{\frac{1}{2^n}e^{t/2}}{n!}x^n\right| = \frac{e^{t/2}}{2^n n!}|x|^n \le \frac{e^{|x|/2}}{2^n n!}|x|^n .$$

Since $\displaystyle\lim_{n\to\infty} \frac{e^{|x|/2}}{2^n n!}|x|^n = e^{|x|/2}\lim_{n\to\infty} \frac{|x|^2}{2^n n!} = 0$ (because $n!$ outgrows any mere exponential), it follows by the Squeeze Theorem that $\displaystyle\lim_{n\to\infty} |R_n(x)| = 0$, and hence also that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$. As previously observed, this means that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n n!} = f(x)$. Note that the particular value of x did not matter … ■