Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2014

Take-home Final Exam
Due on Friday, 18 April, 2014.

Instructions: Do all three of parts ↑, ↑↓, and ↑↓↑, and, if you wish, part ← as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part ↑. Do *all* four (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Define a sequence $\{a_n\}$ by setting $a_0 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for $n \geq 0$. Show that $\lim_{n\to\infty} a_n$ exists and find its value.

2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$ converges or diverges.

3. Let $f_0(x) = x$ and $f_n(x) = \sin(f_{n-1}(x))$ for each $n \geq 1$. Determine the function $f(x)$ such that $f_n \longrightarrow f$ and show that $f_n(x) \underset{\text{unif}}{\longrightarrow} f(x)$ for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

4. Determine the interval of convergence $\displaystyle\sum_{k=0}^{\infty}(k+2)(k+1)x^k = 2 + 6x + 12x^2 + 20x^3 + \cdots$, and find a function $f(x)$ of which it is the Taylor series at 0.

Part ↑↓. Do any *two* (2) of problems **5** – **8**. *[20 = 2 × 10 each]*

5. Show that for any bounded sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers

$$\limsup_{n\to\infty} s_n t_n \leq \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right),$$

and give an example showing that the inequality could be strict.

6. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n c^n$ converges absolutely. Show that the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$. Give an example of such a series and interval for which c is the radius of convergence of the power series.

7. Compute the sum of $\sum_{n=0}^{\infty} \frac{\rho(n)\pi^n}{n!}$, where $\rho(n) = \begin{cases} +1 & n \equiv 0 \text{ or } 1 \pmod 4 \\ -1 & n \equiv 2 \text{ or } 3 \pmod 4 \end{cases}$.

8. Give an example of a series $\displaystyle\sum_{n=0}^{\infty} a_n$ which diverges even though the sequence of partial sums $S_k = \displaystyle\sum_{n=0}^{k} a_n$ has a convergent subsequence, or show that there cannot be such a series.

Part ↑↓↑. Do any *two* (2) of problems **9** – **12**. *[20 = 2 × 10 each]*

9. Suppose $\{q_n\}$ is a sequence listing all the rational numbers in the interval $[0,1]$. Show that for any $r \in [0,1]$, there is a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ such that $\lim_{k\to\infty} q_{n_k} = r$.

10. Suppose $f(x)$ is a function with domain $\mathbb{R}$ which is infinitely differentiable at every point $x \in \mathbb{R}$, and for which there is a constant M such $\left|f^{(n)}(x)\right| \leq M$ for all $n \geq 0$ and $x \in \mathbb{R}$. Show that the Taylor series of $f(x)$ at any point a has radius of convergence $R = \infty$, or find an example of such a function $f(x)$ (and point $a \in \mathbb{R}$) so that $R < \infty$.

11. Suppose the Taylor series of $f(x)$ at 0 is defined, has radius of convergence $R > 0$, and converges to $f(x)$ on $(-R, R)$. If we compute the Taylor series of $f(x)$ at some point $a \neq 0$ such that $-R < a < R$, what is the value of each derivative $f^{(n)}(a)$, for $n \geq 0$, in terms of the derivatives of $f(x)$ at 0?

12. Determine for what values of α and β the series

$$\sum_{k=0}^{\infty}\left(\frac{\alpha}{2k+1} - \frac{\beta}{2k+2}\right) = \frac{\alpha}{1} - \frac{\beta}{2} + \frac{\alpha}{3} - \frac{\beta}{4} + \cdots$$

converges.

Part ←.

↻. Write a poem about real analysis or mathematics in general. *[2]*

[Total = 80]

I HOPE THAT YOU ENJOYED THE COURSE.
HAVE A GREAT SUMMER!