Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Winter 2014

Assignment #11
e is irrational
Due on Friday, 4 April, 2014.

Recall that the Taylor series at 0 of e^x is $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. Let $T_n(x)$ and $R_n(x)$ denote the nth Taylor polynomial and the corresponding remainder term (at 0), *i.e.*

$$\begin{aligned} R_n(x) = e^x - T_n(x) &= e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \\ &= e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \,. \end{aligned}$$

1. Use the integral form of the remainder for a Taylor series (Theorem 7.45 in the text) to show that for $x > 0$, $0 < R_n(x) \le \dfrac{e^x x^{n+1}}{(n+1)!}$. *[2]*

2. Use your estimate for $R_n(x)$ in **1** to show that $0 < R_n(1) < \dfrac{3}{(n+1)!}$. *[1]*

3. Show that e is irrational. *[7]*

Hint: Assume by way of contradiction that $e = \frac{a}{b}$, where a and b are positive integers. Choose an n such that $n > 3b$, and use the "fact" that

$$\frac{a}{b} = e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1)$$

to show that $n!R_n(1)$ must be an integer. Then use **2** to show that it cannot be an integer.