Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2012

Solutions to Assignment #7
Find the limit!

1. Suppose we define a sequence a_n as follows: $a_0 = \frac{1}{2}$ and $a_{n+1} = \dfrac{1}{1+a_n}$ for $n \geq 0$. Show that this sequence converges and find its limit. *[10]*

NOTE: To save the empirically inclined a little effort, here are the first few elements of the sequence [corrected!]:

n	a_n	decimal
0	$\frac{1}{2}$	0.5
1	$\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$	$0.66666666666666\ldots$
2	$\frac{1}{1+\frac{2}{3}} = \frac{3}{5}$	0.6
3	$\frac{1}{1+\frac{3}{5}} = \frac{5}{8}$	0.625
4	$\frac{1}{1+\frac{5}{8}} = \frac{8}{13}$	$0.61538461538461\ldots$
5	$\frac{1}{1+\frac{8}{13}} = \frac{13}{21}$	$0.61904761904761\ldots$
6	$\frac{1}{1+\frac{13}{21}} = \frac{21}{34}$	$0.61764705882352\ldots$
$\vdots$	$\vdots$	$\vdots$

SOLUTION. Suppose $\lim\limits_{n\to\infty} a_n$ exists and $= \alpha$ for some number α. Then

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{1+a_n} = \frac{1}{1+\lim_{n\to\infty} a_n} = \frac{1}{1+\alpha}\,,$$

so α satisfies the equation $\alpha(1+\alpha) = 1$, *i.e.* $\alpha^2 + \alpha - 1 = 0$. Applying the quadratic equation we get that $\alpha = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \dfrac{-1\pm\sqrt{5}}{2}$; that is, α is one of $\dfrac{-1+\sqrt{5}}{2}$ or $\dfrac{-1-\sqrt{5}}{2}$. Since the sequence is composed entirely of non-negative terms, it follows that $\alpha = \dfrac{-1+\sqrt{5}}{2} = 0.6180339887\ldots$.

It remains to show that the sequence does, in fact, converge. We will do this by finding convergent subsequences and verifying that the sequence has the same limit as the subsequences. Our subsequences will be a_{2k}, $k \geq 0$, and a_{2k+1}, $k \geq 0$; we will show these converge by checking that it they are, respectively, an increasing sequence which is bounded above by α and a decreasing sequence which is bounded below by α. We will proceed by induction on $k \geq 0$:

Base step: ($k = 0$ & $k = 2$) $a_0 = \frac{1}{2} < a_2 = \frac{3}{5} < \alpha < a_3 = \frac{5}{8} < a_1 = \frac{2}{3}$.

Inductive hypothesis: Assume that $a_0 < a_2 < \cdots < a_{2k} < \alpha < a_{2k+1} < \cdots < a_3 < a_1$.

Inductive step: By definition,

$$a_{2(k+1)} = a_{2k+2} = \frac{1}{1+a_{2k+1}} = \frac{1}{1+\frac{1}{1+a_{2k}}}$$
$$\text{and} \quad a_{2(k+1)+1} = a_{2k+3} = \frac{1}{1+a_{2k+2}} = \frac{1}{1+\frac{1}{1+a_{2k+1}}}\,.$$

Since $a_{2k-2} < a_{2k} < \alpha < a_{2k+1} < a_{2k-1}$,

$$1+a_{2k-2} < 1+a_{2k} < 1+\alpha < 1+a_{2k+1} < 1+a_{2k-1}\,.$$

It follows that

$$\frac{1}{1+a_{2k-2}} > \frac{1}{1+a_{2k}} > \frac{1}{1+\alpha} = \alpha > \frac{1}{1+a_{2k+1}} > \frac{1}{1+a_{2k-1}}\,,$$

so

$$1+\frac{1}{1+a_{2k-2}} > 1+\frac{1}{1+a_{2k}} > 1+\alpha > 1+\frac{1}{1+a_{2k+1}} > 1+\frac{1}{1+a_{2k-1}}\,,$$

and thus

$$\begin{array}{ccccccccc} \dfrac{1}{1+\frac{1}{1+a_{2k-2}}} & < & \dfrac{1}{1+\frac{1}{1+a_{2k}}} & < & \dfrac{1}{1+\alpha} & < & \dfrac{1}{1+\frac{1}{1+a_{2k+1}}} & < & \dfrac{1}{1+\frac{1}{1+a_{2k-1}}} \\ \| & & \| & & \| & & \| & & \| \\ a_{2k} & & a_{2k+2} & & \alpha & & a_{2k+3} & & a_{2k+1} \end{array}$$

It follows by induction that $a_0 < a_2 < \cdots < a_{2k} < \alpha < a_{2k+1} < \cdots < a_3 < a_1$ for all $k \geq 0$.

By the Monotone Convergence Theorem it now follows that $\lim_{k\to\infty} a_{2k}$ and $\lim_{k\to\infty} a_{2k+1}$ exist and that $\lim_{k\to\infty} a_{2k} = \beta \leq \alpha \leq \gamma = \lim_{k\to\infty} a_{2k+1}$. Note that

$$\beta = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \frac{1}{1+a_{2k-1}} = \frac{1}{1+\gamma}$$
$$\text{and} \quad \gamma = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} \frac{1}{1+a_{2k}} = \frac{1}{1+\beta}\,,$$

so $\beta = \dfrac{1}{1+\gamma} = \dfrac{1}{1+\frac{1}{1+\beta}} = \dfrac{1}{\frac{2+\beta}{1+\beta}} = \dfrac{1+\beta}{2+\beta}$. Thus $1+\beta = \beta(2+\beta) = 2\beta+\beta^2$, so $\beta^2+\beta-1 = 0$. This is the same equation satisfied by α; just as for α, since we are dealing with a limit of non-negative terms, it follows that β is the positive root, *i.e.* $\beta = \dfrac{-1+\sqrt{5}}{2} = \alpha$. A similar argument shows that $\gamma = \alpha$ too. Since both subsequences have the same limit, and they make up the entire sequence between them, the sequence must have the same limit.

Thus $\lim_{n\to\infty} a_n$ exists and $= \dfrac{-1+\sqrt{5}}{2}$. ■

NOTE: The limit of this sequence is $1-\varphi = 1/\varphi$, where φ is the "golden ratio".