Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Winter 2012

Solutions to Assignment #6
More p-tests

1. Determine for which p the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ converges and for which it diverges. *[4]*

Solution. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ converges when $p > 1$, but does not converge when $p \leq 1$, just as in the regular p-Test.

Note that $\displaystyle\frac{\ln(n)}{n^p} > \frac{1}{n^p} > 0$ for $n \geq 3$ since $\ln(n) > 1$ once $n \geq 3 > e$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$, by the p-Test, it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ diverges if $p \leq 1$.

On the other hand, suppose $p > 1$. Then we can write $p = q+r$, where $q > 1$ and $r > 0$. (For example, one could take $r = (p-1)/2$ and $q = (p+1)/2$.) Then $\displaystyle\frac{\ln(n)}{n^p} = \frac{\ln(n)}{n^r} \cdot \frac{1}{n^q}$. Since

$$\lim_{n\to\infty} \frac{\ln(n)}{n^r} = \lim_{x\to\infty} \frac{\ln(x) \to \infty}{x^r \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x^r} \quad \text{(by l'Hôpital's Rule)}$$
$$= \lim_{x\to\infty} \frac{1/n}{rn^{r-1}} = \lim_{x\to\infty} \frac{1}{rn^{r-1}n} = \lim_{x\to\infty} \frac{1 \to 1}{rn^r \to \infty} = 0\,,$$

there is some N such that $\displaystyle\frac{\ln(n)}{n^r} = \left|\frac{\ln(n)}{n^r} - 0\right| < 1$ for all $n \geq N$. Then for all $n \geq N$ we have $\displaystyle\frac{\ln(n)}{n^p} = \frac{\ln(n)}{n^r} \cdot \frac{1}{n^q} < \frac{1}{n^q}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^q}$ converges by the p-Test since $q > 1$, and so it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ converges as well. ■

2. Determine for which p the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^p)}{n^p}$ converges and for which it diverges. *[2]*

Solution. Trick question! $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n^p)}{n^p} = \sum_{n=1}^{\infty} \frac{p\ln(n)}{n^p} = p\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$, so, by **1**, it converges when $p > 1$ and diverges when $p \leq 1$, except for the special case $p = 0$, for which the series converges. (Why?) ■

3. Determine for which p the series $\displaystyle\sum_{n=1}^{\infty} \frac{[\ln(n)]^p}{n^p}$ converges and for which it diverges. *[4]*

SOLUTION. Just like the series in **1**, $\displaystyle\sum_{n=1}^{\infty} \frac{[\ln(n)]^p}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$. In fact, we can verify this using methods similar to those used in the solution to **1**.

First, if $0 \leq p \leq 1$, then $\dfrac{[\ln(n)]^p}{n^p} \geq \dfrac{1}{n^p} > 0$ for $n \geq 3$ since $\ln(n) > 1$ once $n \geq 3 > e$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$, by the p-Test, it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ diverges if $0 \leq p \leq 1$.

Second, if $p < 0$, then $\dfrac{[\ln(n)]^p}{n^p} = \left(\dfrac{\ln(n)}{n}\right)^p = \left(\dfrac{n}{\ln(n)}\right)^{|p|}$, where $|p| > 0$. Note that $\displaystyle\lim_{n\to\infty} \frac{[\ln(n)]^p}{n^p} = \lim_{n\to\infty} \left(\frac{n}{\ln(n)}\right)^{|p|} = \left(\lim_{n\to\infty} \frac{n}{\ln(n)}\right)^{|p|} = \infty$, since $\displaystyle\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{x\to\infty} \frac{x \to \infty}{\ln(x) \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\ln(x)} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$. It then follows that $\displaystyle\sum_{n=1}^{\infty} \frac{[\ln(n)]^p}{n^p}$ does not converge by the Divergence Test.

Third, suppose $p > 1$. As in the solution to **1**, write $p = q + r$, where $q > 1$ and $r > 0$ (*e.g.* $r = (p-1)/2$ and $q = (p+1)/2$). Then $\dfrac{[\ln(n)]^p}{n^p} = \dfrac{[\ln(n)]^p}{n^r} \cdot \dfrac{1}{n^q}$. Note that

$$\lim_{n\to\infty} \frac{[\ln(n)]^p}{n^r} = \lim_{n\to\infty} \left(\frac{\ln(n)}{n^{r/p}}\right)^p = \left(\lim_{n\to\infty} \frac{\ln(n)}{n^{r/p}}\right)^p = \left(\lim_{x\to\infty} \frac{\ln(x)}{x^{r/p}}\right)^p = 0^p = 0\,,$$

because it follows from the fact that $r/p > 0$ that

$$\lim_{x\to\infty} \frac{\ln(x) \to \infty}{x^{r/p} \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x^{r/p}} = \lim_{x\to\infty} \frac{1/x}{\frac{r}{p}x^{r/p-1}} = \lim_{x\to\infty} \frac{1 \quad \to 1}{\frac{r}{p}x^{r/p} \to \infty} = 0\,,$$

with just a little bit of help from l'Hôpital's Rule again. Thus there is some N such that $\dfrac{[\ln(n)]^p}{n^r} = \left|\dfrac{[\ln(n)]^p}{n^r} - 0\right| < 1$ for all $n \geq N$. Then for all $n \geq N$ we have $\dfrac{\ln(n)}{n^p} = \dfrac{[\ln(n)]^p}{n^r} \cdot \dfrac{1}{n^q} < \dfrac{1}{n^q}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^q}$ converges by the p-Test since $q > 1$, and so it follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{[\ln(n)]^p}{n^p}$ converges as well. ■