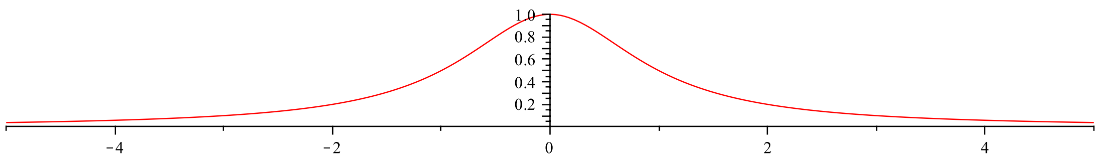
Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Winter 2012

Solution to Assignment #5
Squeezing more out of the Integral Test

Recall that we showed that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverged by interpreting the series as a sum of areas and comparing it to the area under the graph of $f(x) = \dfrac{1}{x}$.

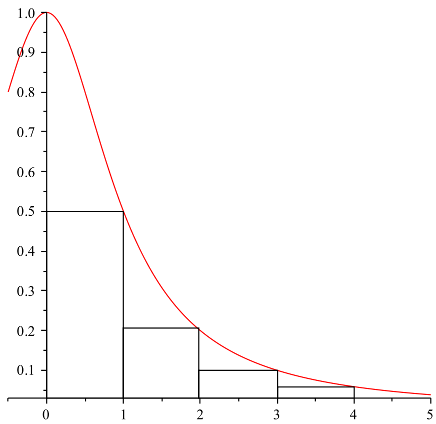
1. Use area-comparison arguments to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges to some number between $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$. *[10]*

Solution. Recall what the graph of $y = \dfrac{1}{1+x^2}$ looks like:



Of course, we'll only need the part where $x \geq 0$ ... :-)

To see that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots$ is less than $\dfrac{\pi}{2}$, consider the areas of the rectangles with base $[n-1, n]$ and height $\dfrac{1}{1+n^2}$ for $n \geq 1$. These fit beneath the curve $y = \dfrac{1}{1+x^2}$ for $0 \leq x < \infty$

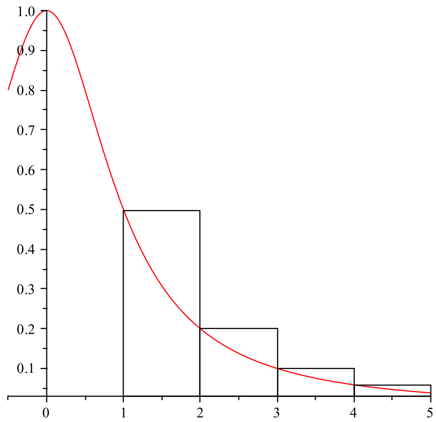


and so their collective area, $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, is less than the area under the curve on $[0, \infty)$. This is given by

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_0^t$$
$$= \lim_{t\to\infty} (\arctan(t) - \arctan(0)) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2},$$

since $\arctan(0) = 0$. Thus $\sum_{n=1}^{\infty} \frac{1}{1+n^2} < \frac{\pi}{2}$.

To see that $\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots$ is greater than $\frac{\pi}{4}$, consider the areas of the rectangles with base $[n, n+1]$ and height $\frac{1}{1+n^2}$ for $n \geq 1$. These enclose all the area beneath the curve $y = \frac{1}{1+x^2}$ for $1 \leq x < \infty$



and so their collective area, which is also $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, is greater than the area under the curve on $[1, \infty)$. This is given by

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_1^t$$
$$= \lim_{t\to\infty} (\arctan(t) - \arctan(1)) = \lim_{t\to\infty} \arctan(t) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

since $\arctan(1) = \frac{\pi}{4}$. Thus $\sum_{n=1}^{\infty} \frac{1}{1+n^2} > \frac{\pi}{4}$ as well. ■