Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2012

Solution to Assignment #2

1. Show that if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$. *[10]*

HINT: This is easier if you first show that if $\lim_{n\to\infty} b_n = M \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

SOLUTION. Following the hint, we first try to show that if $\lim_{n\to\infty} b_n = M \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Suppose $\varepsilon > 0$ is given. We need to find an N such that if $n \geq N$, then $\left|\frac{1}{b_n} - \frac{1}{M}\right| < \varepsilon$. Note that $\left|\frac{1}{b_n} - \frac{1}{M}\right| = \left|\frac{M - b_n}{Mb_n}\right| < \left|\frac{M - b_n}{M \cdot M/2}\right|$ as long as $|M/2| < |b_n| < |3M/2|$. (Larger denominator, smaller fraction ...) Since $\lim_{n\to\infty} b_n = M \neq 0$, there is some N_1 such that whenever $n \geq N_1$, we have $|b_n - M| < M/2$, which amounts to saying that $|M/2| < |b_n| < |3M/2|$. For $n \geq N_1$, therefore, $\left|\frac{1}{b_n} - \frac{1}{M}\right| < \left|\frac{M - b_n}{M \cdot M/2}\right| = |b_n - M| \cdot \frac{2}{M^2}$. Again, since $\lim_{n\to\infty} b_n = M \neq 0$, there is some N_2 such that whenever $n \geq N_2$, we have $|b_n - M| < \varepsilon \cdot M^2/2$. It follows that when $n \geq N = \max\{N_1, N_2\}$ we have

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| < |b_n - M| \cdot \frac{2}{M^2} < \varepsilon \cdot \frac{M^2}{2} \cdot \frac{2}{M^2} = \varepsilon\,,$$

as desired. Thus if $\lim_{n\to\infty} b_n = M \neq 0$, then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

Now suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M \neq 0$. Then, using the limit law for products,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} \frac{1}{b_n}\right) = L \cdot \frac{1}{M} = \frac{L}{M}\,,$$

as desired. ■