Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2012

Take-home Final Exam
Due on Friday, 20 April, 2012.

Instructions: Do all three of parts **A** – **C**, and, if you wish, part ☼ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part A. Do *all* of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Use the $\varepsilon - N$ definition of limits of sequences to verify that $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$.

2. Find the Taylor series of $f(x) = \dfrac{x}{1+x}$ and determine its interval of convergence.

3. Suppose $f_n(x) = \dfrac{x^n}{1+x}$ for each $n \geq 0$ and $f(x) = 0$. Use the definition of uniform convergence to show that $f_n(x) \overset{\text{unif}}{\longrightarrow} f(x)$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

4. How many terms of the Taylor series for $\cos(x)$ about 0, evaluated at $x = \frac{\pi}{2}$, are needed to guarantee that the partial sum is within 0.01 of $\cos\left(\frac{\pi}{2}\right)$?

Part B. Do any *two* (2) of problems **5** – **8**. *[20 = 2 × 10 each]*

5. Give an example of a sequence a_n with subsequences converging to r for every $r \in \mathbb{R}$.

6. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,[\ln(n)]^2}$ converges absolutely, converges conditionally, or diverges.

7. Suppose that $a_n \geq 0$ for each $n \geq 0$, and that $\sum_{n=0}^{\infty} (-1)^n a_n$ converges conditionally. Show that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 1$.

8. Give an example of a series $\sum_{n=0}^{\infty} a_n$ which diverges even though the sequence of partial sums $S_k = \sum_{n=0}^{k} a_n$ has a convergent subsequence.

Part C. Do any *two* (2) of problems **9** – **12**. *[20 = 2 × 10 each]*

9. Suppose that the Taylor series of $f(x)$ has radius of convergence $R = \infty$. Can this series converge uniformly on $\mathbb{R}$ if $f(x)$ is bounded or, respectively, unbounded on $\mathbb{R}$? In each case, either give an example to show that it can or prove that it can't.

10. Suppose t_n, $n \geq 1$, is a sequence such that $\lim_{n\to\infty} t_n$ exists. Show that the sequence is *Cèsaro-summable*, *i.e.* that $\lim_{n\to\infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n}$ exists.

11. Suppose a_n and b_n, $n \geq 0$, are sequences such that *(i)* $0 \leq a_{n+1} \leq a_n$ for all $n \geq 0$, *(ii)* $\lim_{n\to\infty} a_n = 0$, and *(iii)* there is a $B > 0$ such that for all $N \geq 0$, $\left|\sum_{n=0}^{N} b_n\right| \leq B$. Show that $\sum_{n=0}^{\infty} a_n b_n$ converges.

12. Give an example of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ which both converge for all x and such that $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = 1$ for all x, or show that such a pair of series cannot exist.

Part ↻.

↕. Write a poem about real analysis or mathematics in general. *[2]*

[Total = 80]

I hope that you enjoyed this course.
Enjoy the summer!