Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

Quizzes

Quiz #1. ~~Thursday, 23~~ ~~Tuesday, 28~~ Thursday, 30 September, 2010 *(7 minutes)*

1. Show that there is no smallest positive real number. *[5]*

SOLUTION. Suppose (by way of contradiction) that $r > 0$ was the least positive real number. Since the real numbers are a dense linear order, there must be a real number s between 0 and r, *i.e.* such that $0 < s < r$. s is smaller than r and at the same time s is positive. (Remember, $s > 0$!) This contradicts the assumption that r was the smallest positive real number.

Thus there is no smallest positive real number. ■

Quiz #2. Thursday, 30 September, 2010 *(8 minutes)*

1. Show that the sequence $s_n = \dfrac{n-1}{n}$ has a limit. *[5]*

SOLUTION I. Observe that $s_n = \frac{n-1}{n} = \frac{n}{n} - \frac{1}{n} = 1 - \frac{1}{n}$. It follows that s_n is an increasing sequence:

$$n < n+1 \Longrightarrow \frac{1}{n} > \frac{1}{n+1} \Longrightarrow s_n = 1 - \frac{1}{n} < 1 - \frac{1}{n+1} = s_{n+1}$$

Moreover, since $s_n = 1 - \frac{1}{n} < 1$ for every $n \geq 1$, the sequence is bounded above. It follows that $\lim\limits_{n\to\infty} s_n$ exists by the Monotone Convergence Theorem. ■

SOLUTION II. It's pretty easy to guess from the fact that $s_n = \frac{n-1}{n} = \frac{n}{n} - \frac{1}{n} = 1 - \frac{1}{n}$, that $\lim\limits_{n\to\infty} s_n$ ought to be 1. We verify that this is the case.

Suppose $\varepsilon > 0$ is given. We need to find an N such that for all $n \geq N$, we have $|s_n - 1| < \varepsilon$. As usual, we use some reverse engineering to find the N:

$$|s_n - 1| < \varepsilon \Longleftrightarrow \frac{1}{n} = \left|1 - \frac{1}{n} - 1\right| < \varepsilon \Longleftrightarrow n > \frac{1}{\varepsilon}$$

Let N be the least integer (strictly) greater than $\frac{1}{\varepsilon}$. Then if $n \geq N$, we get

$$|s_n - 1| = \left|1 - \frac{1}{n} - 1\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon\,,$$

as desired. Hence $\lim\limits_{n\to\infty} s_n$ exists and equals 1. ■

Quiz #3. ~~Thursday, 7~~ Tuesday, 12 October, 2010 *(10 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges or not. *[5]*

SOLUTION. We will use the (Basic!) Comparison Test. Note that when $n \geq 1$, $n3^n \geq 3^n > 0$, so $0 < \dfrac{1}{n3^n} \leq \dfrac{1}{3^n}$. It follows by the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ does. However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a geometric series with common ratio $0 < \dfrac{1}{3} < 1$, so it converges.

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges. ■

Quiz #4. ~~Thursday, 14~~ Wednesday, 20 October, 2010 *(10 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(5^n)^n}{n!e^n}$ converges or not. *[5]*

SOLUTION. We will use the Ratio Test. Note that since all terms are positive, we can drop the absolute value signs.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{\left(5^{n+1}\right)^{n+1}}{(n+1)!e^{n+1}}}{\frac{(5^n)^n}{n!e^n}} &= \lim_{n\to\infty} \frac{\frac{5^{(n+1)(n+1)}}{(n+1)!e^{n+1}}}{\frac{5^{n\cdot n}}{n!e^n}} \\
&= \lim_{n\to\infty} \frac{5^{n^2+2n+1}}{(n+1)!e^{n+1}} \cdot \frac{n!e^n}{5^{n^2}} \\
&= \lim_{n\to\infty} \frac{5^{2n+1}}{(n+1)e} \\
&\quad \text{Both the numerator and denominator go to infinity,} \\
&\quad \text{so we'll use l'Hôpital's Rule.} \\
&= \lim_{x\to\infty} \frac{5^{2x+1}}{(x+1)e} \\
&= \lim_{x\to\infty} \frac{\ln(5)\cdot 5^{2x+1}\cdot 2}{1\cdot e} \\
&= \frac{2\ln(5)}{e} \lim_{x\to\infty} 5^{2x+1} \\
&= \frac{2\ln(5)}{e} \cdot \infty = \infty > 1
\end{aligned}$$

It follows that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(5^n)^n}{n!e^n}$ diverges. ■

Quiz #5. Thursday, 21 October, 2010 *(15 minutes)*

1. For which values of a does the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a^{2n}+1}$ converge? *[5]*

Solution. This series converges if $|a| > 1$ and diverges if $|a| \leq 1$.

If $|a| > 1$, then $\dfrac{1}{|a|} < 1$, so the geometric series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{|a|}\right)^{2n} = \sum_{n=0}^{\infty} \frac{1}{|a|^{2n}}$ converges. Note that $a^{2n} > 0$ and so $0 < \dfrac{1}{a^{2n}+1} < \dfrac{1}{|a|^{2n}} = \left(\dfrac{1}{|a|}\right)^{2n}$ for all n. It follows by the Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a^{2n}+1}$ converges as well in this case.

If $|a| \leq 1$, then $|a|^{2n} \leq 1$ for all n, so $\displaystyle\lim_{n\to\infty} |a|^{2n} = \lim_{n\to\infty} a^{2n} \leq 1$. It follows that $\displaystyle\lim_{n=0}^{\infty} \frac{1}{a^{2n}+1} = \frac{1}{\left(\lim_{n\to\infty} a^{2n}\right)+1} \geq \frac{1}{1+1} = \frac{1}{2} > 0$, so the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a^{2n}+1}$ diverges by the Divergence Test. *[5]*

Quiz #6. Thursday, 4 November, 2010 *(10 minutes)*

1. Find the Taylor series at $a = 1$ of $f(x) = \ln(x)$. *[5]*

Solution. Observe that if $f(x) = \ln(x)$, then $f'(x) = x^{-1}$, $f''(x) = (-1)x^{-2}$, $f'''(x) = (-1)(-2)x^{-3} = (-1)^2 2! x^{-3}$, $f^{(4)}(x) = (-1)^2 2!(-3)x^{-4} = (-1)^3 3! x^{-4}$, ..., $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$, ... It follows that the Taylor series of $f(x) = \ln(x)$ at $a = 1$ is:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n &= \ln(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!1^{-n}}{n!}(x-1)^n \\ &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \quad \blacksquare\end{aligned}$$

Quiz #7. Thursday, 11 November, 2010 *(15 minutes)*

1. Suppose $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ is a polynomial of degree k. Show that the Taylor series at $a = 0$ of $p(x)$ is equal to $p(x)$. *[5]*

SOLUTION. Consider the derivatives of $p(x)$:

$$\begin{array}{ccc}
n & p^{(n)}(x) & p^{(n)}(0) \\
0 & a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 & 0!a_0 \\
1 & ka_k x^{k-1} + (k-1)a_{k-1}x^{k-2} + \cdots + 2a_2 x + 1a_1 & 1!a_1 \\
2 & k(k-1)a_k x^{k-2} + (k-1)(k-2)a_{k-1}x^{k-2} + \cdots + 2\cdot 1a_2 x & 2!a_2 \\
\vdots & \vdots & \vdots \\
k-1 & k(k-1)\cdots 3\cdot 2a_k x + (k-1)(k-2)\cdots 3\cdot 2\cdot 1a_{k-1} & (k-1)!a_{k-1} \\
k & k(k-1)\cdots 3\cdot 2\cdot 1a_k & k!a_k \\
k+1 & 0 & 0 \\
k+2 & 0 & 0 \\
\vdots & \vdots & \vdots
\end{array}$$

Plugging this into Taylor's formula gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{p^{(n)}(0)}{n!}x^n &= \frac{p^{(0)}(0)}{0!} + \frac{p^{(1)}(0)}{1!}x + \frac{p^{(2)}(0)}{2!}x^2 + \cdots + \frac{p^{(k)}(0)}{k!}x^k + \frac{p^{(k+1)}(0)}{(k+1)!}x^{k+1} + \cdots \\
&= \frac{0!a_0}{0!} + \frac{1!a_1}{1!}x + \frac{2!a_2}{2!}x^2 + \cdots + \frac{k!a_k}{k!}x^k + \frac{0}{(k+1)!}x^{k+1} + \cdots \\
&= a_0 + a_1 x + a_2 x + \cdots + a_k x^k + 0 + \cdots \\
&= p(x)\,,
\end{aligned}$$

as desired. ■

Quiz #8. Thursday, 18 November, 2010 *(15 minutes)*

1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a series with radius of convergence $R > 0$ and $[a,b] \subset (-R,R)$. Why is $f(x)$ bounded on $[a,b]$?

SOLUTION. $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a convergent power series on $(-R,R)$, so it is continuous on $(-R,R)$. Since $[a,b] \subset (-R,R)$, this means that $f(x)$ is continuous on $[a,b]$, and hence is bounded on $[a,b]$ by the max/min stuff from first-year calculus. ■

Quiz #9. ~~Thursday, 25~~ Tuesday, 30 November, 2010 *(15 minutes)*

1. Show that $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$. *[2]*

2. Find the Taylor series at 0 of $\arctan(x) = \displaystyle\int_0^x \frac{1}{1+t^2}\, dt$. *[3]*

Solution to 1. It's a geometric series with first term 1 and common ratio $-x^2$:

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \qquad \blacksquare$$

Solution to 2. We can integrate and differentiate power series term-by-term (within the radius of convergence), so:

$$\begin{aligned}
\arctan(x) &= \int_0^x \frac{1}{1+t^2}\, dt \\
&= \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right) dt \\
&= \int_0^x 1\, dt - \int_0^x t^2\, dt + \int_0^x t^4\, dt - \int_0^x t^6\, dt + \cdots \\
&= t\Big|_0^x - \left.\frac{t^3}{3}\right|_0^x + \left.\frac{t^5}{5}\right|_0^x - \left.\frac{t^7}{7}\right|_0^x + \cdots \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\end{aligned}$$

This is the Taylor series of $\arctan(x)$ at 0 by the uniqueness of Taylor series: any power series equal to a function on a non-trivial interval must be the function's Taylor series. $\blacksquare$

Quiz #10. Thursday, 2 December, 2010 *(15 minutes)*

1. Recall that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$ for $x \in (-1,1)$. Show that the series cannot converge uniformly to $\dfrac{1}{1-x}$ over the whole interval $(-1,1)$.

   Hint: $\dfrac{1}{1-x}$ has an asymptote at $x = 1$.

Solution. For the series $\displaystyle\sum_{n=0}^{\infty} x^n$ to converge uniformly to $\dfrac{1}{1-x}$ over the whole interval $(-1,1)$ would mean, by definition, that for any $\varepsilon > 0$, there is some N such that for all $n \geq N$ and all $x \in (-1,1)$, $\left| \dfrac{1}{1-x} - \displaystyle\sum_{k=0}^{n} x^k \right| < \varepsilon$.

However, if we have any $\varepsilon > 0$ and N whatsoever, and $x \in (-1,1)$, $\displaystyle\sum_{k=0}^{N} x^k$ is bounded above by $\sum_{k=0}^{N} 1^k = \sum_{k=0}^{N} 1 = N+1$. Since $\displaystyle\lim_{x \to 1^-} \frac{1}{1-x} = \infty$, we can find an $x < 1$ close enough to 1 to ensure that $\dfrac{1}{1-x} > N + 1 + \varepsilon > \displaystyle\sum_{k=0}^{N} x^k + \varepsilon$, so that $\left| \dfrac{1}{1-x} - \displaystyle\sum_{k=0}^{n} x^k \right| > \varepsilon$.

Thus the definition of uniform convergence must fail if applied to the convergence of $\displaystyle\sum_{n=0}^{\infty} x^n$ to $\dfrac{1}{1-x}$ over the whole interval $(-1,1)$. ■

Quiz #11. Thursday, 2 December, 2010 *(15 minutes)*

1. Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely. Show that $\sum_{n=0}^{\infty} a_n \cos(nx)$ converges for all x.

Solution. Since $|\cos(nx)| \leq 1$ for all x, $|a_n \cos(nx)| \leq |a_n|$ for all x and n. Since $\sum_{n=0}^{\infty} a_n$ converges absolutely, it follows by the Comparison Test that $\sum_{n=0}^{\infty} a_n \cos(nx)$ converges absolutely, and hence converges, for all x. ■