Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

Solutions to Assignment #5
Alternatives?

Recall that the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, diverges, while its close relation, the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converges.

1. Determine whether the following relative of the harmonic series converges or diverges: $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{\beta(n)}{n}$, where $\beta : \mathbb{N}^+ \to \{-1, 1\}$ is given by $\beta(n) = \begin{cases} +1 & n \neq 0 \pmod 3 \\ -1 & n = 0 \pmod 3 \end{cases}$. *[3]*

SOLUTION. It diverges. To see this we group the terms of the series in threes and combine them:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\beta(n)}{n} &= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots \\
&= \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8} - \frac{1}{9}\right) + \cdots \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{3k+1} + \frac{1}{3k+2} - \frac{1}{3k+3}\right) \\
&= \sum_{k=0}^{\infty} \frac{(3k+2)(3k+3) + (3k+1)(3k+3) - (3k+1)(3k+2)}{(3k+1)(3k+2)(3k+3)} \\
&= \sum_{k=0}^{\infty} \frac{9k^2 + 15k + 6 + 9k^2 + 12k + 3 - 9k^2 - 9k - 2}{27k^3 + 54k^2 + 90k + 6} \\
&= \sum_{k=0}^{\infty} \frac{9k^2 + 18k + 7}{27k^3 + 54k^2 + 90k + 6}
\end{aligned}$$

The individual terms of the last form above are clearly rational functions with the degree of the numerator being 2 and the degree of the denominator being 3. Since $3 - 2 = 1 \not> 1$, it follows by the Generalized p-Test that the series diverges. ■

2. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{\tau(n)}{n} x^n$, where $\tau : \mathbb{N}^+ \to \{-1, 1\}$ is some function that assigns one of ± 1 to each $n > 0$. *[3]*

SOLUTION. As usual, we will use the Ratio Test to find the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{\tau(n+1)}{n+1}x^{n+1}}{\frac{\tau(n)}{n}x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\tau(n+1)}{\tau(n)}\cdot\frac{n}{n+1}\cdot x\right| \\
&= \lim_{n\to\infty}\left|\frac{\tau(n+1)}{\tau(n)}\right|\cdot\left|\frac{n}{n+1}\right|\cdot|x| \\
&= \lim_{n\to\infty}\frac{|\tau(n+1)|}{|\tau(n)|}\cdot\frac{n}{n+1}\cdot|x| \\
&= \lim_{n\to\infty}\frac{1}{1}\cdot\frac{n}{n+1}\cdot|x| \\
&= |x|\lim_{n\to\infty}\frac{n}{n+1} \\
&= |x|\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\
&= |x|\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} \\
&= |x|\cdot\frac{1}{1+0} \\
&= |x|
\end{aligned}$$

It follows by the Ratio Test that the series will converge (absolutely) when $|x| < 1$ and diverge when $|x| > 1$, so the radius of convergence is $R = 1$. ■

3. Find an example of a function $\tau : \mathbb{N}^+ \to \{-1, 1\}$ which makes the interval of convergence of $\sum_{n=1}^{\infty} \frac{\tau(n)}{n} x^n$ be, respectively,

a. $(-R, R)$ **b.** $[-R, R)$ **c.** $(-R, R]$ **d.** $[-R, R]$

(where R is the radius of convergence from **2**) or show that there is no such τ. *[4]*

SOLUTION. Recall from **2** that $R = 1$. It is possible to find suitable functions $\tau : \mathbb{N}^+ \to \{-1, 1\}$ to make the interval of convergence of $\sum_{n=1}^{\infty} \frac{\tau(n)}{n} x^n$ be each of the four alternatives.

b. This is probably the simplest case. Let $\tau(n) = 1$ for $n \geq 1$; then the series in question is $\sum_{n=1}^{\infty} \frac{x^n}{n}$. This diverges at $x = 1$, since $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is just the harmonic series, which we know diverges (by, say, the p-Test). It converges at $x = -1$, since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is just the alternating harmonic series, which we know converges (conditionally, by the Alternating Series Test). The interval of convergence is therefore $[-R, R)$ for this example. □

c. This is almost as simple. Let $\tau(n) = (-1)^n$ for $n \geq 1$; then the series in question is $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$. This diverges at $x = -1$, since $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is just the harmonic series, which we know diverges (by, say, the p-Test). It converges at $x = 1$, since $\sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is just the alternating harmonic series, which we know converges (conditionally, by the Alternating Series Test). The interval of convergence is therefore $(-R, R]$ for this example. $\square$

d. Here we have to start working a little harder. Let $\tau(n) = \begin{cases} +1 & n = 1 \text{ or } 2 \pmod 4 \\ -1 & n = 3 \text{ or } 0 \pmod 4 \end{cases}$; then the series in question is:

$$\sum_{n=1}^{\infty} \frac{\tau(n) x^n}{n} = x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} - \frac{x^7}{7} - \frac{x^8}{8} + \cdots$$

We will show that this converges for both $x = 1$ and $x = -1$, using tricks similar to those used in the solution to **1**.

First,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tau(n) 1^n}{n} &= 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{4k+1} + \frac{1}{4k+2} - \frac{1}{4k+3} - \frac{1}{4k+4} \right) \\
&= \sum_{k=1}^{\infty} \frac{\begin{array}{c}(4k+2)(4k+3)(4k+4) + (4k+1)(4k+3)(4k+4) \\ - (4k+1)(4k+2)(4k+4) - (4k+1)(4k+2)(4k+3)\end{array}}{(4k+1)(4k+2)(4k+3)(4k+4)} \\
&= \sum_{k=1}^{\infty} \frac{64k^2 + 80k + 22}{256k^4 + 640k^3 + 560k^2 + 200k + 24}
\end{aligned}$$

The individual terms of the last form above are clearly rational functions with the degree of the numerator being 2 and the degree of the denominator being 4. Since $4 - 2 = 2 > 1$, it follows by the Generalized p-Test that the series converges absolutely.

Second, and very similarly,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tau(n)(-1)^n}{n} &= -1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \cdots \\
&= \sum_{k=1}^{\infty} \left(-\frac{1}{4k+1} + \frac{1}{4k+2} + \frac{1}{4k+3} - \frac{1}{4k+4} \right) \\
&= \sum_{k=1}^{\infty} \frac{\begin{array}{c}-(4k+2)(4k+3)(4k+4) + (4k+1)(4k+3)(4k+4) \\ + (4k+1)(4k+2)(4k+4) - (4k+1)(4k+2)(4k+3)\end{array}}{(4k+1)(4k+2)(4k+3)(4k+4)} \\
&= \sum_{k=1}^{\infty} \frac{-16k - 10}{256k^4 + 640k^3 + 560k^2 + 200k + 24}
\end{aligned}$$

The individual terms of the last form above are clearly rational functions with the degree of the numerator being 1 and the degree of the denominator being 4. Since $4 - 1 = 3 > 1$, it follows by the Generalized p-Test that the series converges absolutely.

The interval of convergence is therefore $[-R, R]$ for this example. $\square$

a. Let $\tau(n) = \begin{cases} +1 & n \neq 0 \pmod 4 \\ -1 & n = 0 \pmod 4 \end{cases}$; then the series in question is:

$$\sum_{n=1}^{\infty} \frac{\tau(n)x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \cdots$$

We will show that this diverges for both $x = 1$ and $x = -1$, using methods similar to those used in the solution for **d** above.

First,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tau(n)1^n}{n} &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{4k+1} + \frac{1}{4k+2} + \frac{1}{4k+3} - \frac{1}{4k+4} \right) \\
&= \sum_{k=1}^{\infty} \frac{\begin{array}{r} (4k+2)(4k+3)(4k+4) + (4k+1)(4k+3)(4k+4) \\ + (4k+1)(4k+2)(4k+4) - (4k+1)(4k+2)(4k+3) \end{array}}{(4k+1)(4k+2)(4k+3)(4k+4)} \\
&= \sum_{k=1}^{\infty} \frac{128k^3 + 288k^2 + 192k + 38}{256k^4 + 640k^3 + 560k^2 + 200k + 24}
\end{aligned}$$

The individual terms of the last form above are clearly rational functions with the degree of the numerator being 3 and the degree of the denominator being 4. Since $4 - 3 = 1 \not> 1$, it follows by the Generalized p-Test that the series diverges.

Second, and similarly,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tau(n)(-1)^n}{n} &= -1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots \\
&= \sum_{k=1}^{\infty} \left(-\frac{1}{4k+1} + \frac{1}{4k+2} - \frac{1}{4k+3} - \frac{1}{4k+4} \right) \\
&= \sum_{k=1}^{\infty} \frac{\begin{array}{r} -(4k+2)(4k+3)(4k+4) + (4k+1)(4k+3)(4k+4) \\ - (4k+1)(4k+2)(4k+4) - (4k+1)(4k+2)(4k+3) \end{array}}{(4k+1)(4k+2)(4k+3)(4k+4)} \\
&= \sum_{k=1}^{\infty} \frac{-128k^3 - 224k^2 - 128k - 26}{256k^4 + 640k^3 + 560k^2 + 200k + 24}
\end{aligned}$$

The individual terms of the last form above are clearly rational functions with the degree of the numerator being 3 and the degree of the denominator being 4. Since $4 - 3 = 1 \not> 1$, it follows by the Generalized p-Test that the series diverges.

The interval of convergence is therefore $(-R, R)$ for this example. □

It follows that all four alternatives are possible. In the solutions for **d** and **a** above, the polynomials were otained with the help of the program `Yacas` ("Yet Another Computer Algebra System"). ■

Bonus. Determine, as best you can, whether the following relative of the harmonic series converges or diverges: $\sum_{n=1}^{\infty} \frac{\rho(n)}{n}$, where $\rho : \mathbb{N}^+ \to \{-1, 1\}$ randomly chooses one of ± 1 for each $n > 0$, in such a way that $\lim_{n\to\infty} \frac{\rho(1)+\rho(2)+\cdots+\rho(n)}{n} = 0$. (That is, the series is "alternating on average.") *[1]*

NON-SOLUTION. Only one person tried this, and didn't get too far, so the deadline for this problem has been extended to Tuesday, 21 December, 2010, and the number of possible points has been increased to *[5]* from *[1]*. ■